Ramsey's Theorem on Trees

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20 September, 2016

Reverse Mathematics

- Main Question of Reverse Mathematics: What are the appropriate axioms for mathematics?
- History: 1970's, Harvey Friedman and Stephen Simpson.
- Standard Reference: Subsystems of Second Order Arithmetic, by Simpson.

Reverse Mathematics

- Language: The Language of Second Order Arithmetic.
- Model: $\langle \mathcal{M}, S \rangle$ is a model of Second Order Arithmetic.
  - $\mathcal{M}$ is a model of First Order Arithmetic.
    - We use ω to denote the standard model of arithmetic.
    - $\mathcal{M}$ may not be standard.
  - $S \subseteq P(\mathcal{M})$.
- Axioms:
  - Usual axioms of Peano Arithmetic (PA), where the induction is restricted to Σ^0_1 formulas
  - Set Existence Axioms.

Inductions in Reverse Mathematics

- Big Five:
  - $\mathsf{RCA}_0 \Leftarrow \mathsf{WKL}_0 \Leftarrow \mathsf{ACA}_0 \Leftarrow \mathsf{ATR}_0 \Leftarrow \Pi^1_1\text{-}\mathsf{CA}_0$
- $\mathsf{WKL}_0 \restriction$ First Order $= \Sigma^0_1$ Induction; $\mathsf{ACA}_0 \restriction$ First Order $=$ PA.
- Induction:

$$\forall x(\forall y < x\, \phi(y) \Rightarrow \phi(x)) \Rightarrow \forall x\, (\phi(x))$$

  - If ϕ is restricted to Σ^0_n formulas, then the induction is called Σ^0_n Induction (Denoted as $I\Sigma^0_n$, or $I\Sigma_n$ for short.)
  - Similarly, we have $I\Pi_n$, $I\Delta_n$.
- Main Question on Induction: What are the appropriate inductions for mathematics?

Inductions Axioms

- Bounding:

$$\forall y < x\,(\exists w\,\phi(y,w)) \Rightarrow \exists b\,(\forall y < x\,\exists w < b\,\phi(y,w))$$

  - If ϕ is restricted to Σ_n^0 formulas, then the bounding is called Σ_n^0 Bounding (Denoted as $B\Sigma_n^0$, or $B\Sigma_n$ for short.)
  - Similarly, we have $B\Pi_n$, $B\Delta_n$.

Theorem (Kirby and Paris)

- $I\Sigma_n \Leftrightarrow I\Pi_n$
- $B\Pi_n \Leftrightarrow B\Delta_{n+1} \Leftrightarrow B\Sigma_{n+1}$
- $I\Sigma_n \Rightarrow B\Sigma_n$, $B\Sigma_{n+1} \Rightarrow I\Sigma_n$, $B\Sigma_n \not\Rightarrow I\Sigma_n$

Ramsey's Theorem

- $X, H \subseteq \mathcal{M}$.
- Let $[X]^n$ be the collection of all subsets of X of size n.
- Coloring $C : [\mathcal{M}]^n \to k$.
- Homogenous set H: $C \upharpoonright [H]^n$ is a constant function.

Theorem (Ramsey)

Suppose $k, n \geq 1$. Every coloring $C : [\mathcal{M}]^n \to k$ has an infinite homogenous set.

- Notation:
  - k, n are fixed. RT^n_k.
  - n is fixed. $\mathrm{RT}^n = \forall k\ \mathrm{RT}^n_k$.

Ramsey's Theorem on Trees

- $2^{<m}$: Collection of all (M-finite) binary strings of length $< m$.
- $2^{<\mathcal{M}}$: Collection of all (M-finite) binary strings in $\mathcal{M}$.
- $X, H \subseteq 2^{<\mathcal{M}}$.
- Let $[X]^n$ be the collection of all **compatible** subsets of X of size n.
- Coloring $C : \left[2^{<\mathcal{M}}\right]^n \to k$.
- Homogenous/Monochromatic tree H: $H \cong 2^{<m}$ (Order Isomorphic, $m \in \mathcal{M} \bigcup \{\mathcal{M}\}$) and $C \upharpoonright [H]^n$ is a constant function.

Theorem

Suppose $k, n \geq 1$. Every coloring $C : \left[2^{<\mathcal{M}}\right]^n \to k$ has an infinite monochromatic tree.

Ramsey's Theorem on Trees

- Notation:
  - k, n are fixed. TT^n_k.
  - n is fixed. $\mathsf{TT}^n = \forall k\ \mathsf{TT}^n_k$.
- $\mathsf{TT}^n_k \Rightarrow \mathsf{RT}^n_k$

TT v.s. RT

Theorem (Logicians)

Axiom	First Order	Second Order (Over RCA_0)
TT^1	$> B\Sigma_2, \leq I\Sigma_2$	$> RCA_0 + B\Sigma_2, \perp WKL_0, < ACA_0$
RT^1	$B\Sigma_2$	$RCA_0 + B\Sigma_2$
TT^2_2	$\geq B\Sigma_2, \leq I\Sigma_3$	$> RT^2_2, < ACA_0$
RT^2_2	$\geq B\Sigma_2, < I\Sigma_2$	$> RCA_0 + B\Sigma_2, \perp WKL_0, < ACA_0$
$TT^n_k, n \geq 3, k \geq 2$	PA	ACA_0
RT^n_k	PA	ACA_0

TT^1 Assuming $I\Sigma_2$

- $\mathsf{TT}^1 \Rightarrow \mathsf{RT}^1 \Rightarrow B\Sigma_2$.
- $I\Sigma_2 \Rightarrow \mathsf{TT}^1$
  - Given $C : \left[2^{<\mathcal{M}}\right] \to k$.
  - Consider the maximal $c_0 < k$ such that $\exists\sigma\forall\tau \supseteq \sigma(C(\tau) \geq c_0)$.
  - σ_0 is a witness for the c_0.
  - c_0 is dense among extensions of σ_0.
  - The monochromatic tree is recursive.

Question

- Assume $B\Sigma_2 + \neg I\Sigma_2$ and $C : [2^{<\mathcal{M}}] \to k$.
- Is there an infinite monochromatic tree?
- What is the complexity of an monochromatic tree? Is there a monochromatic tree preserving $B\Sigma_2$?

Density

Theorem (Corduan, Groszek and Mileti)

Suppose $\mathcal{M} \models B\Sigma_2 + \neg I\Sigma_2$. *There is* $k \in \mathcal{M}$ *with a recursive* $C : \left[2^{<\mathcal{M}}\right] \to k$ *such that there is no recursive monochromatic tree.*

Corollary

- $RCA_0 + B\Sigma_2 \not\vdash TT^1$.
- *In that coloring* C*, every color is nowhere dense.*

Lowness

Theorem (Chong, Li, Wang and Yang)

Suppose $\mathcal{M} \models B\Sigma_2 + \neg I\Sigma_2$. There is $k \in \mathcal{M}$ with a recursive $C : [2^{<\mathcal{M}}] \to k$ such that there is no $\mathbf{0}'$-recursive monochromatic tree.

Corollary

- $WKL_0 + B\Sigma_2 \not\vdash TT^1$.
- *In that coloring C, no monochromatic tree is low.*

Existence

Theorem (Chong, Li, Wang and Yang)

Suppose $\mathcal{M} \models B\Sigma_2 + \neg I\Sigma_2$ and $C : [2^{<\mathcal{M}}] \to k$ is recursive. There is a regular monochromatic tree.

- A set X is regular, if $X \cap \mathcal{M}$-finite $= \mathcal{M}$-finite.
- Non-definable solution.

Complexity

Theorem (Chong, Li, Wang and Yang)

Suppose $\mathcal{M} \models B\Sigma_2 + \neg I\Sigma_2$.

- *There is* $k \in \mathcal{M}$ *with a recursive* $C : [2^{<\mathcal{M}}] \to k$ *such that there is no recursive monochromatic tree but there is a low monochromatic tree.*
- *There is* $k \in \mathcal{M}$ *with a recursive* $C : [2^{<\mathcal{M}}] \to k$ *such that there is no low monochromatic tree but there is a monochromatic tree preserving* $B\Sigma_2$.

Conjecture

$\mathsf{TT}^1 \not\vdash I\Sigma_2$.

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Thank you.