Randomness notions in Muchnik and Medvedev degrees

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Motivation

Main Question

- Could we construct a more random set from a given random set?
- How to formalize? Why important?

Computable

- Logicians … computable by a Turing machine
- Mathematicians … a formula can be simplified such as 2+3, 2x+1+4x, some integration, etc.
- Statisticians and data scientists … computable with random access

With random access

- Which sets are computable with random access?
- An old answer: computable sets

Old answer

Theorem (De Leeuwe, Moore, Shannon, Shapiro (1956), Sacks)**.** *If A is not computable, then the class*

$$\{X \in 2^\omega \ : \ A \leq_T X\}$$

has measure 0*.*

So, if a set is computable with random access, then the set should be computable. The story is over, in this case.

One variant is the case of poly-time computability, which is the famous question of BPP = P?

If there many answers,

- Problem: Construct some non-computable set.
- Without random access: Impossible.
- With random access: Possible.
- How difficult is it to compute a set in a given class?

Definition. Let $P, Q \subseteq 2^\omega$. We say that P is Muchnik reducible to Q, denoted by $P \leq_w Q$, if, for every $f \in Q$, there exists $g \in P$ such that $g \leq_T f$.

Loosely speaking, any element in Q can compute some element in P.

Definition. Let $P, Q \subseteq 2^\omega$. We say that P is Medvedev reducible to Q, denoted by $P \leq_s Q$, if there exists a Turing functional Φ such that $\Phi^f \in P$ for every $f \in Q$.

The difference is uniformity.

	non-uniform	uniform
functional	reverse math	Weihrauch degree
class	Muchnik degree	Medvedev degree

Theorem (Simpson 2004).

$$2^\omega <_w \mathrm{MLR} <_w \mathrm{PA}$$

where

- MLR *is the class of all ML-random sets,*
- PA *is the class of consistent complete extensions of Peano arithmetic.*

The class of random sets seems natural examples in Muchnik degrees.

Theorem (from algorithmic randomness).

$$\mathrm{WR} \Longleftarrow \mathrm{SR} \Longleftarrow \mathrm{CR} \Longleftarrow \mathrm{MLR}$$

and

$$\begin{tikzcd}
 & & \mathrm{W2R} \arrow[dl, Rightarrow] & \\
\mathrm{MLR} & \mathrm{DiffR} \arrow[l, Rightarrow] & & \mathrm{2R} \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \\
 & & \mathrm{DemR} \arrow[ul, Rightarrow] &
\end{tikzcd}$$

Every arrow is strict.

Theorem (Muchnik degrees).

$$\begin{tikzcd} \mathrm{WR} & \mathrm{SR} \arrow[l] \arrow[r, equal] & \mathrm{CR} & \mathrm{MLR} \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} & & \mathrm{W2R} \arrow[dl] & \\ \mathrm{MLR} \arrow[r, equal] & \mathrm{DiffR} & & \mathrm{2R} \arrow[ul] \arrow[dl] \\ & & \mathrm{DemR} \arrow[ul] & \end{tikzcd}$$

- We ask whether each arrow is strict. This can be interpreted as we ask whether we can construct a more random set from a given random set.
- In particular, we look at how uniformity plays a role in this setting.

Proof

Theorem.

$$\mathrm{CR} <_w \mathrm{MLR}$$

Proof. Suppose MLR $<_w$ CR for a contradiction.

There exists a high minimal degree **a** by Cooper '73.

Then, there exists a computably random set $X \in \mathbf{a}$, because every high degree contains a computably random set by Nies, Stephan, and Terwijn '05.

By the assumption there exists a ML-random set $Y \leq_T X$. Since **a** is minimal and Y can not be computable, we have $Y \equiv_T X$. Thus, the Turing degree of Y is minimal.

However, any ML-random degree can not be minimal by van Lambalgen's theorem. □

Theorem.

$$\mathrm{SR} \equiv_w \mathrm{CR}$$

Proof. Every Schnorr random set can compute a computably random set, because

(i) if the Schnorr random set is not high, then it is already ML-random,

(ii) if the Schnorr random set is high, then it computes a computably random set.

$\square$

Rather non-uniform proof!

Theorem.

$$\mathrm{MLR} \equiv_w \mathrm{DiffR}$$

Proof. Every ML-random set can compute a difference random. Let $X \oplus Y$ be a ML-random set.

(i) If $X \geq_T \emptyset'$, then Y is 2-random, thus difference random.

(ii) If $X \not\geq_T \emptyset'$, then X is difference random.

$\square$

Again, non-uniform proof.

Theorem.

$$\mathrm{MLR} <_s \mathrm{DiffR}$$

Theorem.

$$\mathrm{SR} <_s \mathrm{CR}$$

$X \in 2^\omega$ is not computably random if (and only if) $M(X \upharpoonright n) = \infty$ for some computable martingale M.

$X \in 2^\omega$ is not Schnorr random if and only if $M(X \upharpoonright f(n)) > n$ for infinitely many n for some computable order f and some computable martingale M.

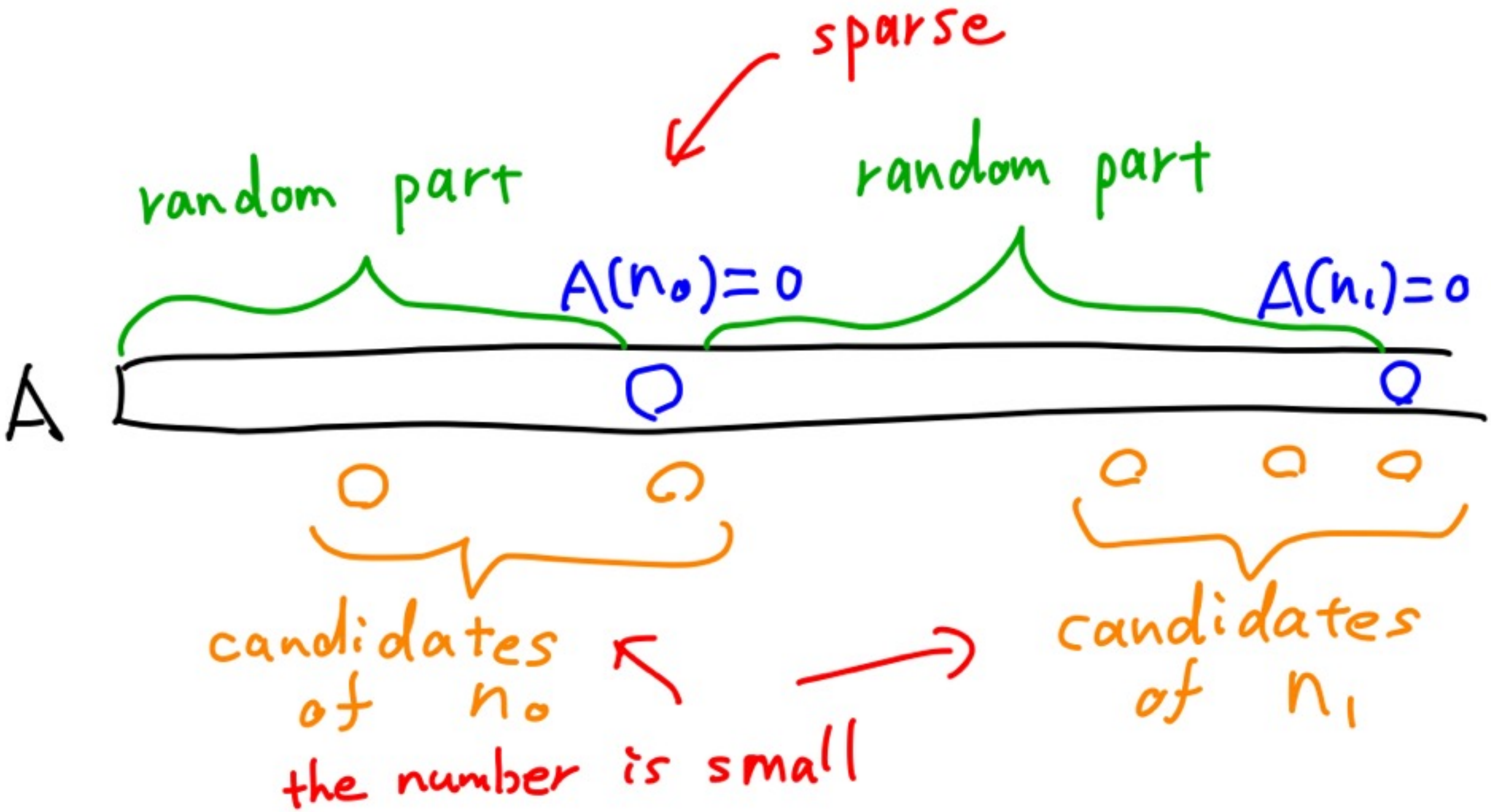
The difference between CR and SR is the rate of divergence.

CR $\not\leq_s$ SR means that, for every Turing functional Φ, there exists $A \in$ SR such that $\Phi^A \notin$ CR.

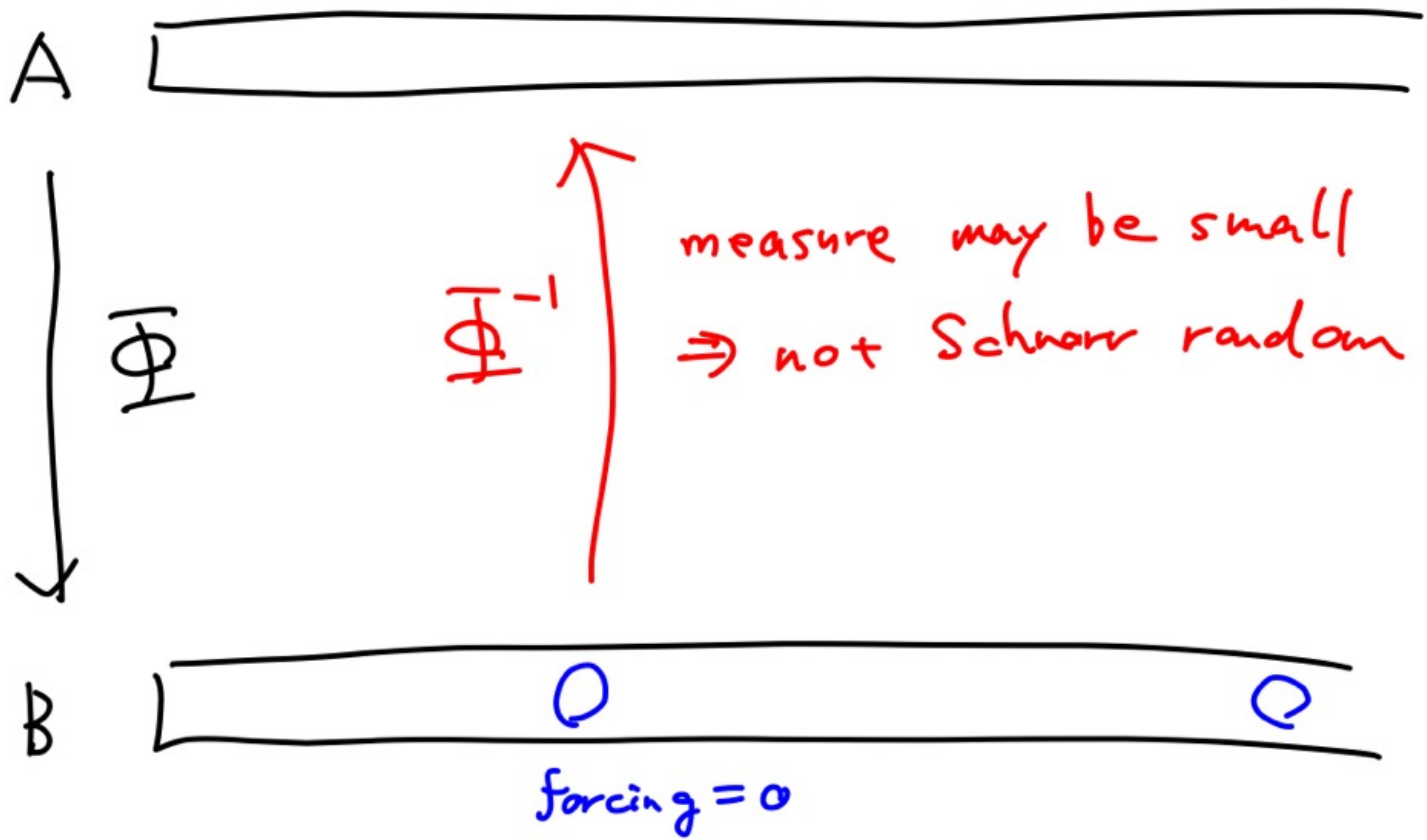
When $\Phi = \mathrm{id}$, it means that there exists $A \in$ SR such that $A \notin$ CR.

In fact we extend the method of separating SR and CR.

- Construct a random set A
- Forcing A(n_k)=0 in sparse positions
  => too sparse not to be Schnorr random
- Number of candidates of n_k is small
  => so small that some computable martingale succeeds (very slowly)


sparse
random part
random part
A(n₀)=0
A(n₁)=0
A
0
0
0
0
0
0
0
candidates of n₀
candidates of n₁
the number is small

- Construct A in SR and B=Phi(A) not in CR
- Forcing B(n_k)=0 in some positions
- Number of candidates of n_k should be small
- However, measure of inverse image may be too small (may be empty) and some computable martingale may succeed in Schnorr sense even if n_k is very sparse


A
Φ
Φ^{-1}
measure may be small
$\Rightarrow$ not Schnorr random
B
forcing = 0

- Induced measure is “close to” uniform measure
=> The same method can be applied

- Induced measure is “far from” uniform measure
=> The another method will be applied

Let $\Phi :\subseteq 2^\omega \to 2^\omega$ be a.e. computable function. Then, the induced measure μ is defined by

$$\mu(\sigma) = \lambda(\{X \in 2^\omega \ : \ \Phi(X) \in [\sigma]\}).$$

The measure μ is computable.

The dividing condition is

Case 1 $\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$

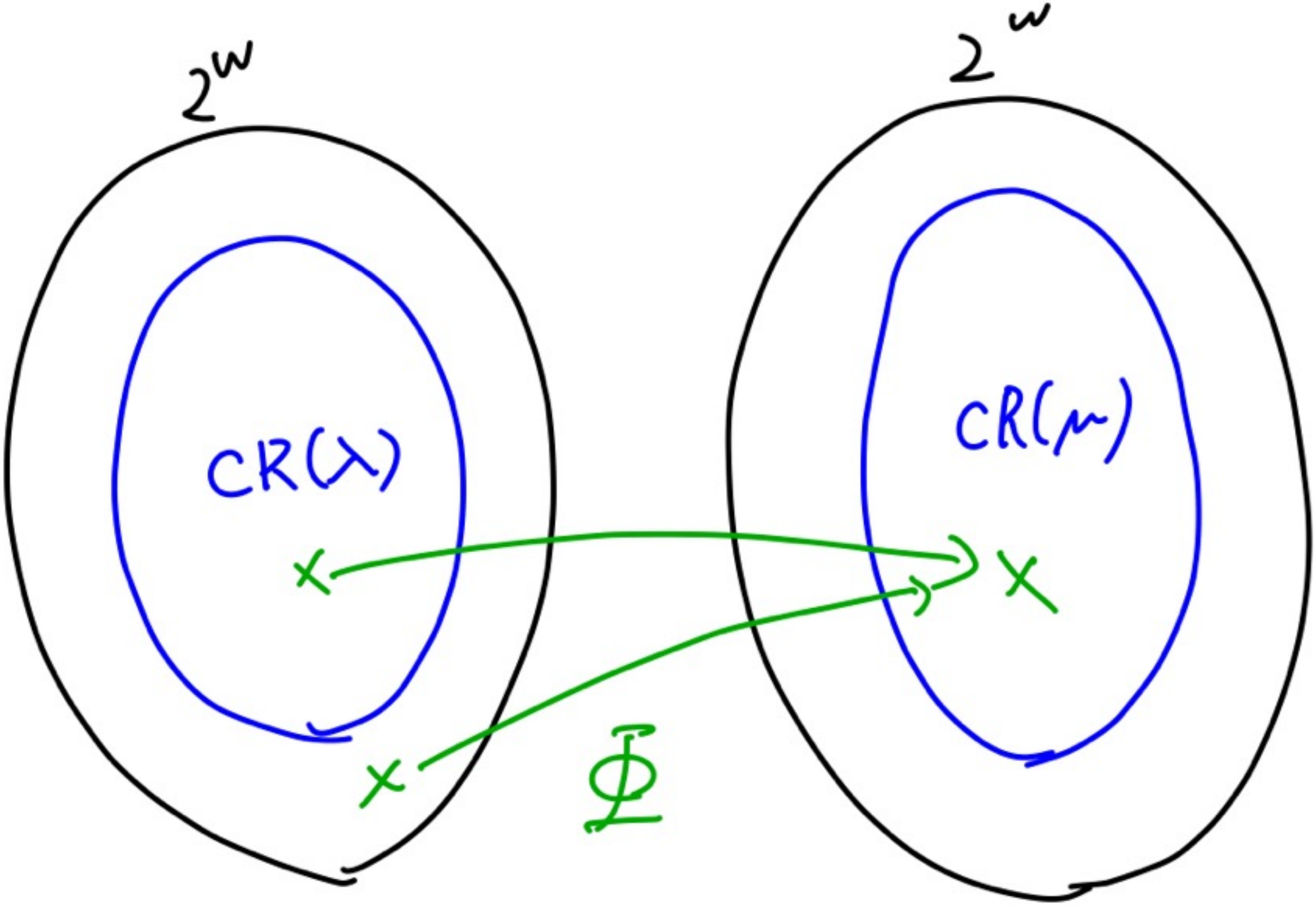
Case 2 $\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda)$

Case 2: $\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda)$

Proof. There exists $Y \in \mathrm{CR}(\mu) \setminus \mathrm{CR}(\lambda)$.

By the no-randomness-from-nothing result for computable randomness by Rute, there exists $X \in \mathrm{CR}(\lambda)$ such that $\Phi(X) = Y$.

Then, $X \in \mathrm{SR}$ and $\Phi(X) \notin \mathrm{CR}$. $\square$



2^ω
2^ω
$CR(\lambda)$
$CR(\mu)$
Φ

Case 1: $\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$

Lemma. *Let μ, ν be computable measures. Then, we have*

$$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\nu) \Rightarrow \mathrm{MLR}(\mu) \subseteq \mathrm{MLR}(\nu) \Rightarrow \nu \ll \mu.$$

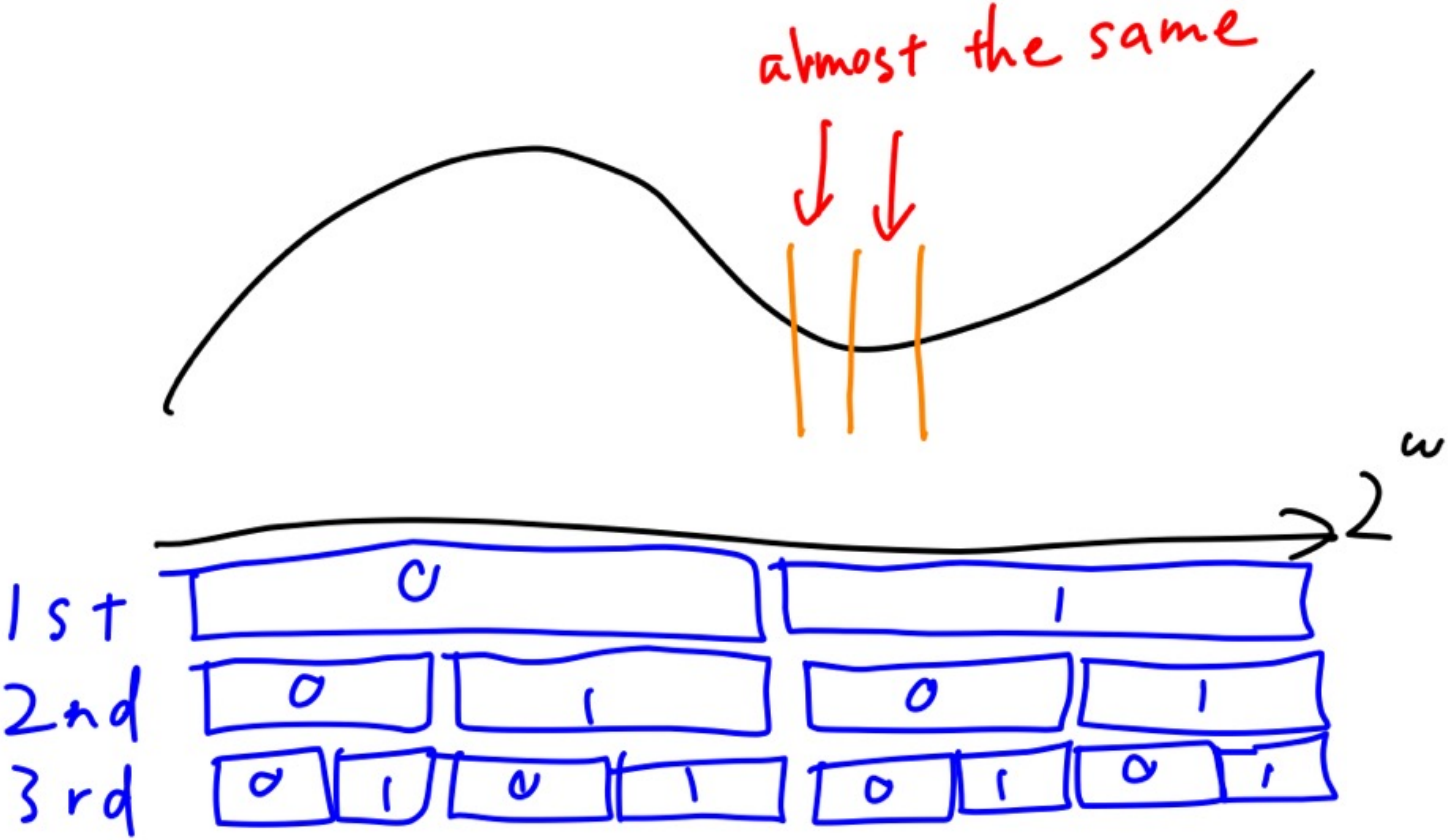
Here, $\ll$ means absolute continuity.

Case 1: $\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$

Lemma. *Let* $\Phi :\subseteq 2^\omega \to 2^\omega$ *be an a.e. computable function. Let* μ *be the measure induced from* Φ *and* λ*. Assume that* $\lambda \ll \nu$*. Then, for each* $\sigma \in 2^{<\omega}$*, we have*

$$\lim_{n\to\infty} \lambda\{X \in [\sigma] \ : \ \Phi(X)(n) = 0\} = \frac{1}{2}\lambda(\sigma).$$

Proof. By the Radon-Nikodym theorem and Lévy's zero-one law. $\square$


almost the same
2^w
1st
0
1
2nd
0
1
0
1
3rd
0
1
0
1
0
1
0
1

Summary

- We studied randomness notions in Muchnik degrees and Medvedev degrees. They are related to reverse maths and Weihrauch degrees.
- We found two problems that is possible non-uniformly but impossible uniformly.
- Interesting interaction between analysis and computability.