Reverse mathematics and the strong Tietze extension theorem

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A conjecture of Giusto & Simpson (2000)

Conjecture (Giusto & Simpson)

The following are equivalent over RCA_0:

(1) WKL_0.

(2) *Let* $\widehat{X}$ *be a compact complete separable metric space, let* C *be a closed subset of* $\widehat{X}$, *and let* $f\colon C \to \mathbb{R}$ *be a continuous function with a modulus of uniform continuity. Then there is a continuous function* $F\colon \widehat{X} \to \mathbb{R}$ *with a modulus of uniform continuity such that* $F \upharpoonright C = f$.

(3) *Same as* (2) *with 'closed' replaced by 'closed and separably closed.'*

(4) *Special case of* (2) *with* $\widehat{X} = [0,1]$.

(5) *Special case of* (3) *with* $\widehat{X} = [0,1]$.

Let $\mathsf{sTET}_{[0,1]}$ denote statement (5).

Definitions in RCA_0 (metric spaces)

Let's remember what all the words in the conjecture mean in RCA_0.

A real number is coded by a sequence $\langle q_k : k \in \mathbb{N}\rangle$ of rationals such that $\forall k \forall i(|q_k - q_{k+i}| \leq 2^{-k})$.

A complete separable metric space $\widehat{A}$ is coded by a non-empty set A and a metric $d\colon A \times A \to \mathbb{R}^{\geq 0}$.

A point in $\widehat{A}$ is coded by a sequence $\langle a_k : k \in \mathbb{N}\rangle$ of members of A such that $\forall k \forall i(d(a_k, a_{k+i}) \leq 2^{-k})$.

A complete separable metric space $\widehat{A}$ is compact if there are finite sequences $\langle\langle x_{i,j} : j \leq n_i\rangle : i \in \mathbb{N}\rangle$ with each $x_{i,j} \in \widehat{A}$ such that

$$(\forall z \in \widehat{A})(\forall i \in \mathbb{N})(\exists j \leq n_i)(d(x_{i,j}, z) < 2^{-i}).$$

Definitions in RCA_0 (the interval [0,1])

The interval $[0,1]$ is a complete separable metric space coded by the set $\{q \in \mathbb{Q} : 0 \leq q \leq 1\}$ (with the usual metric).

The sequence $\langle\langle j2^{-i} : j \leq 2^i\rangle : i \in \mathbb{N}\rangle$ witnesses that $[0,1]$ is compact according to the definition on the previous slide.

So RCA_0 proves that $[0,1]$ is a compact complete separable metric space.

Contrast this to the following facts (Friedman):

- The Heine-Borel compactness of $[0,1]$ is equivalent to WKL_0 over RCA_0.
- The sequential compactness of $[0,1]$ is equivalent to ACA_0 over RCA_0.

Definitions in RCA_0 (closed and separably closed)

An open set in a metric space $\widehat{A}$ is coded by a set $U \subseteq \mathbb{N} \times A \times \mathbb{Q}^{>0}$ (thought of as an enumeration of open balls).

A point $x \in \widehat{A}$ is in the open set coded by U if $(\exists \langle n, a, r \rangle \in U)(d(x, a) < r)$.

A closed set in a metric space is the complement of an open set.

A separably closed set in a metric space $\widehat{A}$ is coded by a sequence $\langle x_n : n \in \mathbb{N} \rangle$ of points in $\widehat{A}$. A point $x \in \widehat{A}$ is in the separably closed set if $(\forall q \in \mathbb{Q}^{>0})(\exists n \in \mathbb{N})(d(x, x_n) < q)$.

In RCA_0, a closed set need not be separably closed, and a separably closed set need not be closed.

In ACA_0, a subset of a compact metric space is closed if and only if it is separably closed. Both implications require ACA_0 (Brown).

Definitions in RCA_0 (continuous functions)

A continuous partial function from a metric space $\widehat{A}$ to a metric space $\widehat{B}$ is coded by a set $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^{>0} \times B \times \mathbb{Q}^{>0}$ (thought of as an enumeration of pairs of open balls $\mathcal{B}(a, r)$ and $\mathcal{B}(b, s)$).

If the pair $\langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle$ is enumerated, it means that every element of $\mathcal{B}(a, r)$ is mapped into the closure of $\mathcal{B}(b, s)$

The enumeration must satisfy:

- If $\langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle$ and $\langle \mathcal{B}(a, r), \mathcal{B}(b', s') \rangle$ are enumerated, then $\mathcal{B}(b, s) \cap \mathcal{B}(b', s') \neq \emptyset$.
- $\langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle$ is enumerated and $\mathcal{B}(a', r') \subseteq \mathcal{B}(a, r)$, then $\langle \mathcal{B}(a', r'), \mathcal{B}(b, s) \rangle$ is enumerated.
- $\langle \mathcal{B}(a, r), \mathcal{B}(b, s) \rangle$ is enumerated and $\mathcal{B}(b, s) \subseteq \mathcal{B}(b', s')$, then $\langle \mathcal{B}(a, r), \mathcal{B}(b', s') \rangle$ is enumerated.

Definitions in RCA_0 (continuous functions)

A point $x \in \widehat{A}$ is in the domain of the function coded by Φ if

$$(\forall \epsilon > 0)(\Phi \text{ lists some } \langle \mathcal{B}(a,r), \mathcal{B}(b,s)\rangle \text{ with } x \in \mathcal{B}(a,r) \text{ and } s < \epsilon),$$

in which case the value of the function at x is the $y \in \widehat{B}$ such that $y \in \overline{\mathcal{B}(b,s)}$ for every enumerated $\langle \mathcal{B}(a,r), \mathcal{B}(b,s)\rangle$ with $x \in \mathcal{B}(a,r)$.

Today we mostly care about functions that are piecewise constant and whose domains are unions of disjoint closed intervals.

Definitions in RCA_0 (modulus of uniform continuity)

A modulus of uniform continuity for a continuous function $f\colon \widehat{A} \to \widehat{B}$ is a function $h\colon \mathbb{N} \to \mathbb{N}$ such that

$$(\forall n \in \mathbb{N})(\forall x, y \in \widehat{A})(d(x,y) < 2^{-h(n)} \to d(f(x), f(y)) < 2^{-n}).$$

Over RCA_0, the following are equivalent (Brown, Simpson):

- WKL_0.
- Every continuous function on a compact complete separable metric space has a modulus of uniform continuity.
- Every continuous function on $[0,1]$ has a modulus of uniform continuity.

Also, in RCA_0, a continuous function $f\colon [0,1] \to \mathbb{R}$ has a modulus of uniform continuity if and only if it has a Weierstraß approximation (more on this later).

Remember the conjecture?

Conjecture (Giusto & Simpson)

The following are equivalent over RCA_0:

(1) WKL_0.

(2) *Let* $\widehat{X}$ *be a compact complete separable metric space, let* C *be a closed subset of* $\widehat{X}$, *and let* $f\colon C \to \mathbb{R}$ *be a continuous function with a modulus of uniform continuity. Then there is a continuous function* $F\colon \widehat{X} \to \mathbb{R}$ *with a modulus of uniform continuity such that* $F \restriction C = f$.

(3) *Same as* (2) *with 'closed' replaced by 'closed and separably closed.'*

(4) *Special case of* (2) *with* $\widehat{X} = [0,1]$.

(5) *Special case of* (3) *with* $\widehat{X} = [0,1]$ ($\mathsf{sTET}_{[0,1]}$).

Need (1) $\Rightarrow$ (2) and $\mathsf{sTET}_{[0,1]} \Rightarrow$ (1).

The Tietze extension theorem in RCA_0

If we give up on uniform continuity, then the Tietze extension theorem is provable in RCA_0.

Theorem (in RCA_0; Brown, Simpson)

Let $\widehat{X}$ be a complete separable metric space, let C be a closed subset of $\widehat{X}$, and let $f\colon C \to [a,b] \subseteq \mathbb{R}$ be a continuous function. Then there is a continuous function $F\colon \widehat{X} \to [a,b]$ such that $F \restriction C = f$.

This immediately gives the strong Tietze extension theorem in WKL_0 (i.e., (1) $\Rightarrow$ (2) on the previous slide) because in WKL_0, continuous functions on compact spaces have moduli of uniform continuity.

The strong Tietze extension theorem for located sets

Giusto & Simpson obtained a version of the strong Tietze extension theorem in RCA_0 by assuming that the closed set C is also located.

A closed or separably closed subset C of a metric space $\widehat{X}$ is located if there is a continuous distance function $f\colon \widehat{X} \to \mathbb{R}$ such that $(\forall x \in \widehat{X})(f(x) = \inf\{d(x,y) : y \in C\})$.

Theorem (in RCA_0; Giusto & Simpson)

Let $\widehat{X}$ be a compact complete separable metric space, let C be a closed and located subset of $\widehat{X}$, and let $f\colon C \to \mathbb{R}$ be a continuous function with a modulus of uniform continuity. Then there is a continuous function $F\colon \widehat{X} \to \mathbb{R}$ with a modulus of uniform continuity such that $F \restriction C = f$.

Digression on located sets

The following is in the context of a compact metric space $\widehat{X}$. $\widehat{X}$ may be taken to be $[0,1]$. All results are due to Giusto & Simpson.

RCA_0 proves the following:

- If $C \subseteq \widehat{X}$ is closed and located, then it is separably closed.
- If $C \subseteq \widehat{X}$ is separably closed and located, then it is closed.

The following are equivalent to ACA_0 over RCA_0:

- Every closed $C \subseteq \widehat{X}$ is separably closed.
- Every separably closed $C \subseteq \widehat{X}$ is closed.
- Every closed $C \subseteq \widehat{X}$ is located.
- Every separably closed $C \subseteq \widehat{X}$ is located.

Over RCA_0, WKL_0 is equivalent to "every closed and separably closed $C \subseteq \widehat{X}$ is located."

Strong Tietze extension theorems for separably closed sets

This version is without uniform continuity.

Theorem (Giusto & Simpson)

The following are equivalent over RCA_0:

(1) ACA_0.

(2) *Let* $\widehat{X}$ *be a compact complete separable metric space, let* C *be a separably closed subset of* $\widehat{X}$, *and let* $f\colon C \to \mathbb{R}$ *be a continuous function. Then there is a continuous function* $F\colon \widehat{X} \to \mathbb{R}$ *such that* $F \restriction C = f$.

(3) *Special case of* (2) *with* $\widehat{X} = [0,1]$.

Strong Tietze extension theorems for separably closed sets

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Theorem (Giusto & Simpson)

The following are equivalent over RCA_0:

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2. *Let* $\widehat{X}$ *be a compact complete separable metric space, let* C *be a separably closed subset of* $\widehat{X}$, *and let* $f\colon C \to \mathbb{R}$ *be a continuous function with a modulus of uniform continuity. Then there is a continuous function* $F\colon \widehat{X} \to \mathbb{R}$ *with a modulus of uniform continuity such that* $F \restriction C = f$.
3. *Special case of* (2) *with* $\widehat{X} = [0, 1]$.

Brass tacks

Remember one more time that $\mathsf{sTET}_{[0,1]}$ is the following statement:

Let C be a closed and separably closed subset of $[0,1]$, and let $f\colon C \to \mathbb{R}$ be a continuous function with a modulus of uniform continuity. Then there is a continuous function $F\colon [0,1] \to \mathbb{R}$ with a modulus of uniform continuity such that $F \restriction C = f$.

We want to show that $\mathsf{RCA}_0 + \mathsf{sTET}_{[0,1]} \vdash \mathsf{WKL}_0$.

First, we give Giusto & Simpson's proof that $\mathsf{RCA}_0 \nvdash \mathsf{sTET}_{[0,1]}$.

They show that REC is not a model of $\mathsf{sTET}_{[0,1]}$ by building C and f to diagonalize against every possible Weierstraß approximation of an extension F of f.

Weierstraß approximations in RCA_0

In RCA_0, having a modulus of uniform continuity is the same as having a Weierstraß approximation.

Theorem (in RCA_0; Simpson)

If $F\colon [0,1] \to \mathbb{R}$ is continuous, then F has a modulus of uniform continuity if and only if there is a sequence of polynomials with rational coefficients $\langle p_n : n \in \mathbb{N} \rangle$ such that

$$(\forall n \in \mathbb{N})(\forall x \in [0,1])(|F(x) - p_n(x)| < 2^{-n}).$$

So we want to define recursive codes for a C and an $f\colon C \to \mathbb{R}$ with a modulus of uniform continuity such that there is no recursive Weierstraß approximation to an extension.

Preparing the domain

For each $e \in \omega$, let

$$I_e = \left[\frac{1}{2^{2e+1}}, \frac{1}{2^{2e}}\right].$$

Let $D = \{0\} \cup \bigcup_{e \in \omega} I_e$.

We shrink D to C by taking advantage of the fact that $[0,1]$ (and every I_e) is not Heine-Borel compact in REC.

Fix an enumeration of an open covering of $[0,1] \cap$ REC that has no finite sub-covering.

Translate this covering to each I_e by the appropriate linear function.

Shrinking I_e and defining f

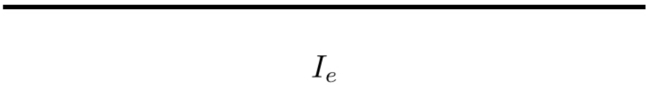
Plan: On interval I_e, diagonalize against Φ_e computing a Weierstraß approximation for an extension of f.

Implement the following strategy on I_e:

- Let $\langle (a_k, b_k) : k \in \omega \rangle$ enumerate the open cover of $I_e \cap \mathsf{REC}$ with no finite sub-cover.
- Enumerate the intervals (a_k, b_k) into the complement of C while waiting for $\Phi_e(2e+1)$ to converge.
- If $\Phi_e(2e+1)\downarrow = p(x)$, stop and choose $q \in I_e \cap \mathbb{Q}$ not yet covered.
- If $p(q) \leq 0$, define $f(x) = 2^{-2e}$ on what's left of I_e. Otherwise define $f(x) = -2^{-2e}$ on what's left of I_e.
- If $\Phi_e(2e+1)\uparrow$, then I_e is erased and we don't need to define f there.
- f has modulus of uniform continuity $n \mapsto 2n+2$.

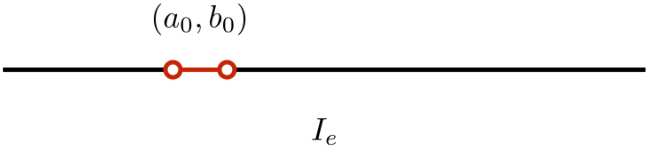
Shrinking I_e and defining f

Start with I_e:



Shrinking I_e and defining f

Delete (a_0, b_0):



Shrinking I_e and defining f

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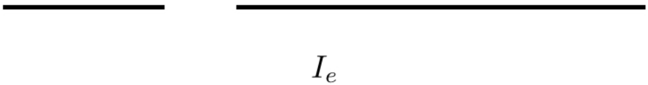
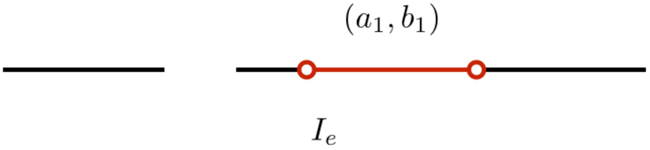


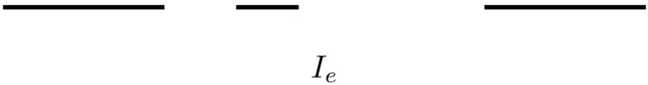
Shrinking I_e and defining f

Delete (a_1, b_1):



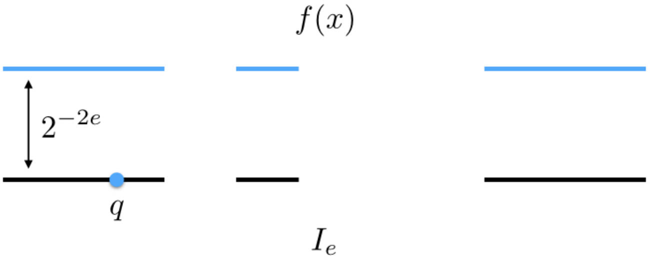
Shrinking I_e and defining f

Delete (a_1, b_1):



Shrinking I_e and defining f

If $\Phi_e(2e+1)\downarrow = p(x)$, chose q and define f on I_e (here $p(q) \leq 0$):



$\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \mathsf{sTET}_{[0,1]} \vdash \mathsf{WKL}_0$

Let $g_0, g_1 \colon \mathbb{N} \to \mathbb{N}$ be injections with disjoint ranges. We want to find a separating set.

It would be nice if we could do what we did before:

- Use I_e to code whether or not e should be in the separating set.
- Chip away at I_e until we see $e \in \operatorname{ran} g_0$ or $e \in \operatorname{ran} g_1$.
- If $e \in \operatorname{ran} g_0$ (g_1), let $f(x) = 2^{-2e}$ (-2^{-2e}) on the remainder of I_e.
- If e is not in $\operatorname{ran} g_0$ or $\operatorname{ran} g_1$, then I_e is disjoint from $\operatorname{dom} f$.

(Note that this plan uses $\neg\mathsf{WKL}_0$.)

Let $F \colon [0,1] \to \mathbb{R}$ be an extension with modulus of uniform continuity H.

If we knew a point q within $2^{-H(2e+2)}$ of a point in $I_e \cap C$, then we could decide whether or not to put e in the separating set.

$\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \mathsf{sTET}_{[0,1]} \vdash \mathsf{WKL}_0$

Let $F\colon [0,1] \to \mathbb{R}$ be an extension with modulus of uniform continuity H.

If we knew a point q within $2^{-H(2e+2)}$ of a point in $I_e \cap C$, then we could decide whether or not to put e in the separating set:

- Use F to find a rational $2^{-(2e+2)}$-approximation r of $F(q)$.
- If $e \in \operatorname{ran} g_0$, there is $x \in I_e \cap C$ within $2^{-H(2e+2)}$ of q such that $F(x) = 2^{-2e}$.
- This means $F(q)$ is within $2^{-(2e+2)}$ of 2^{-2e}.
- So r is within $2^{-(2e+1)}$ of 2^{-2e}. So $r > 0$.
- Put e in the separating set if $r > 0$. Otherwise leave e out.

But how would you find q?

Reorganizing the pre-domain

We reorganize f's pre-domain to arrange the q's ahead of time.

Replace I_e with infinitely many disjoint closed intervals $\langle I_{e,m} : m \in \mathbb{N}\rangle$ contained in the old I_e.

Ensure each $I_{e,m}$ has length at most 2^{-m}.

Fix an open cover of $I_{e,m}$ with no finite subcover (and ensure that the cover of $I_{e,m}$ doesn't intersect a different $I_{e',m'}$).

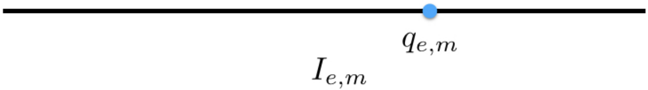
Choose $q_{e,m} \in I_{e,m}$ for each $e, m \in \mathbb{N}$.

Now run the plan from two slides ago on each $I_{e,m}$:

- Chip away at $I_{e,m}$ until we see $e \in \operatorname{ran} g_0$ or $e \in \operatorname{ran} g_1$.
- If $e \in \operatorname{ran} g_0$ (g_1), let $f(x) = 2^{-2e}$ (-2^{-2e}) on the remainder of $I_{e,m}$.
- If e is not in $\operatorname{ran} g_0$ or $\operatorname{ran} g_1$, then every $I_{e,m}$ is disjoint from $\operatorname{dom} f$.

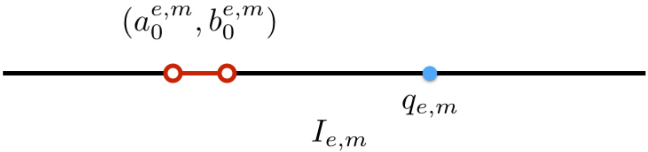
Shrinking $I_{e,m}$ and defining f

Start with $I_{e,m}$ and $q_{e,m}$:



Shrinking $I_{e,m}$ and defining f

Delete $(a_0^{e,m}, b_0^{e,m})$:



Shrinking $I_{e,m}$ and defining f

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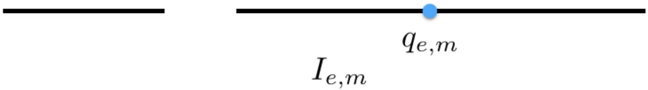
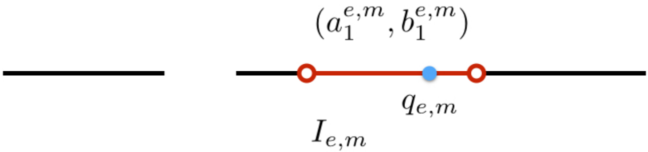


Shrinking $I_{e,m}$ and defining f

Delete $(a_1^{e,m}, b_1^{e,m})$:



Shrinking $I_{e,m}$ and defining f

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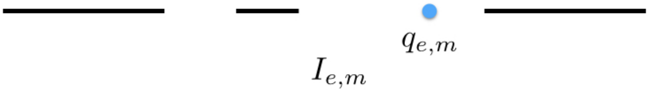
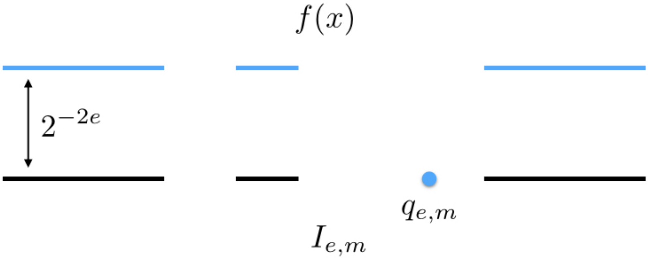


Shrinking $I_{e,m}$ and defining f

If $e \in \operatorname{ran} g_0$, define f to be 2^{-2e} on $I_{e,m}$:



A lemma to find the $I_{e,m}$'s

Lemma (in $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0$)

For each $e \in \mathbb{N}$, let $I_e = [2^{-(2e+1)}, 2^{-2e}]$. There are pairwise disjoint closed intervals with rational endpoints $\langle I_{e,m} : e, m \in \mathbb{N}\rangle$, rationals $\langle q_{e,m} : e, m \in \mathbb{N}\rangle$, and open intervals with rational endpoints $\langle (a_k^{e,m}, b_k^{e,m}) : e, m, k \in \mathbb{N}\rangle$ such that

(i) $\{0\} \cup \bigcup_{e,m \in \mathbb{N}} I_{e,m}$ *is closed;*

(ii) $q_{e,m} \in I_{e,m}$*;*

(iii) $I_{e,m} \subseteq I_e$*, and the length of* $I_{e,m}$ *is less than* 2^{-m}*;*

(iv) $\langle (a_k^{e,m}, b_k^{e,m}) : k \in \mathbb{N}\rangle$ *is an open cover of* $I_{e,m}$ *with no finite subcover;*

(v) *if* $\langle e, m\rangle \neq \langle e', m'\rangle$*, then* $I_{e,m}$ *and* $(a_k^{e',m'}, b_k^{e',m'})$ *are disjoint.*

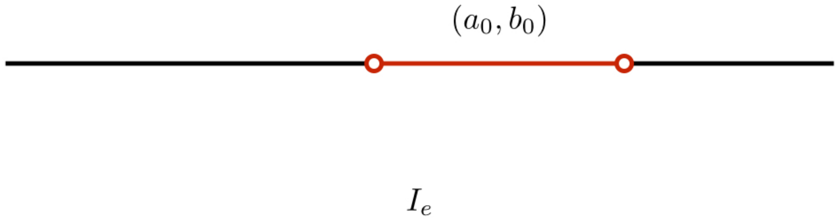
Finding the $I_{e,m}$'s

Start with I_e:

I_e

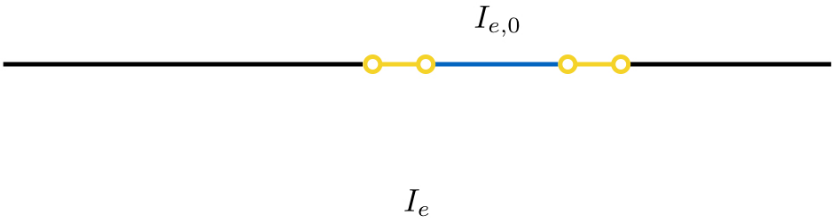
Finding the $I_{e,m}$'s

Look at the first interval of a cover with no finite subcover:



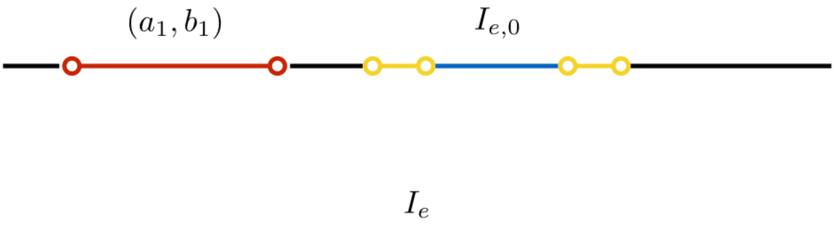
Finding the $I_{e,m}$'s

Choose $I_{e,0}$ to be a closed interval in (a_0, b_0) of length less than 2^{-0}:



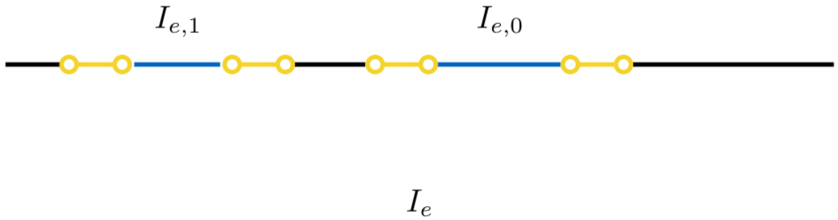
Finding the $I_{e,m}$'s

Look at the second interval of a cover with no finite subcover:



Finding the $I_{e,m}$'s

Choose $I_{e,1}$ to be a closed interval in (a_1, b_1) of length less than 2^{-1}:



The strong Tietze extension theorem and WKL_0

Theorem

The following are equivalent over RCA_0:

1. WKL_0.
2. *Let* $\widehat{X}$ *be a compact complete separable metric space, let* C *be a closed subset of* $\widehat{X}$, *and let* $f\colon C \to \mathbb{R}$ *be a continuous function with a modulus of uniform continuity. Then there is a continuous function* $F\colon \widehat{X} \to \mathbb{R}$ *with a modulus of uniform continuity such that* $F \restriction C = f$.
3. *Same as* (2) *with 'closed' replaced by 'closed and separably closed.'*
4. *Special case of* (2) *with* $\widehat{X} = [0,1]$.
5. *Special case of* (3) *with* $\widehat{X} = [0,1]$.

Thank you!

Thank you for coming to my talk!
Do you have a question about it?