# Recent Work on Weakly Represented Families and Reverse Mathematics


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# The Papers

HJS 2016: Rupert Hölzl, Sanjay Jain and Frank Stephan. Inductive Inference and Reverse Mathematics. Annals of Pure and Applied Logic, on Journal Homepage; was also at STACS 2015.

HRSZ 2017: Rupert Hölzl, Dilip Raghavan, Frank Stephan and Jing Zhang. Weakly Represented Families in Reverse Mathematics. Rod Downey Festschrift, to appear in 2017.

CHRS 2018: C.T. Chong, Rupert Hölzl, Dilip Raghavan and Frank Stephan. Induction and Independence in Reverse Mathematics. Paper under preparation.

# Models of Reverse Mathematics

A model $(\mathbf{M}, \mathbf{S}, +, \cdot, \mathbf{0}, \mathbf{1}, =, \in)$ of reverse mathematics consists of a first order part $\mathbf{M}$ which is similar but not necessarily equal to the set $\mathbb{N}$ of natural numbers and a second order part $\mathbf{S}$ of subsets of $\mathbf{M}$ which satisfy certain closure and induction properties like being closed under computations relative to oracles in $\mathbf{S}$. Although formally $\mathbf{S}$ consists of subsets of $\mathbf{M}$, informally one also includes function from $\mathbf{M}^{\mathbf{k}}$ to $\mathbf{M}$ into $\mathbf{S}$ and assumes that they are coded in some natural way as sets.

All models satisfy $\mathbf{RCA_0}$ and $\mathbf{RCA_0}$ is assumed throughout this talk everywhere.

The letters $\mathbf{M}$ and $\mathbf{S}$ will in the following always refer to the corresponding parts of the model of reverse mathematics currently considered.

# Induction Axioms

$\mathbf{I\Sigma^0_n}$: For every partial $\mathbf{\Sigma^0_n}$-function $\mathbf{F}$ and every $\mathbf{a}$, the set $\{\mathbf{F(b) : b \leq a}\}$ has a maximum.

$\mathbf{B\Sigma^0_n}$: For every total $\mathbf{\Sigma^0_n}$-function $\mathbf{F}$ and every $\mathbf{a}$, the set $\{\mathbf{F(b) : b \leq a}\}$ has an upper bound.

Alternative formalisations go with induction over $\mathbf{\Sigma^0_n}$ formulas.

The following ordering is given:
$\mathbf{I\Sigma^0_1} \leftarrow \mathbf{B\Sigma^0_2} \leftarrow \mathbf{I\Sigma^0_2} \leftarrow \mathbf{B\Sigma^0_3} \leftarrow \mathbf{I\Sigma^0_3} \leftarrow \ldots$; no arrow can be reversed.
Note that $\mathbf{RCA_0} \vdash \mathbf{I\Sigma^0_1}$ by definition.

If the model satisfies $\mathbf{M} = \mathbb{N}$ (so called $\omega$-models) then all induction axioms hold by definition.

# Weakly Represented Families

Informally, a weakly represented family of functions is given by an oracle $\mathbf{A} \in \mathbf{S}$ and a partial-recursive $\psi$ such that $\psi_{\mathbf{e}}^{\mathbf{A}}$ is in the family iff it is total; let $\mathbf{E}$ denote the index set of the total members $\psi_{\mathbf{e}}^{\mathbf{A}}$ which does not need to be in $\mathbf{S}$. A family is called uniformly represented iff $\mathbf{E} \in \mathbf{S}$; in this way one can avoid partial functions. A weakly represented family of sets is given by the weakly represented family of the characteristic functions of these sets.

Weakly represented families are interesting since they permit to formalise various concepts from the study of cardinal invariants or recursion theory in reverse mathematics.

Dzhafarov and Mummert [2013] considered the more general notion of enumerated families of sets.

# Axioms on Model Strength

The axiom $\mathbf{DOM}$ says that for every weakly represented family $\{\mathbf{f_e} : \mathbf{e} \in \mathbf{E}\}$ of functions there is a function $\mathbf{g} \in \mathbf{S}$ which grows faster than every $\mathbf{f_e}$.

The axiom $\mathbf{HI}$ says that for every weakly represented family $\{\mathbf{f_e} : \mathbf{e} \in \mathbf{E}\}$ of functions there is a function $\mathbf{g} \in \mathbf{S}$ such that no function $\mathbf{f_e}$ grows faster than $\mathbf{g}$.

The axiom $\mathbf{AVOID}$ says that for every weakly represented family $\{\mathbf{f_e} : \mathbf{e} \in \mathbf{E}\}$ of functions there is a function $\mathbf{g} \in \mathbf{S}$ such that no function $\mathbf{f_e}$ agrees with $\mathbf{g}$ on an infinite domain.

The axiom $\mathbf{MEET}$ says that for every weakly represented family $\{\mathbf{f_e} : \mathbf{e} \in \mathbf{E}\}$ of functions there is a function $\mathbf{g} \in \mathbf{S}$ such that every function $\mathbf{f_e}$ agrees with $\mathbf{g}$ on an infinite domain.

The axiom $\mathbf{BI}$ says that for every weakly represented family $\{\mathbf{A_e} : \mathbf{e} \in \mathbf{E}\}$ of infinite sets there is a set $\mathbf{B} \in \mathbf{S}$ such that no $\mathbf{A_e}$ is a subset of $\mathbf{B}$ or disjoint to $\mathbf{B}$.

# MEET and HI

Theorem [HRSZ 2017]: $\mathbf{MEET}$ and $\mathbf{HI}$ are equivalent.

Proof. Assume that $\{\mathbf{f_e} : \mathbf{e} \in \mathbf{E}\}$ is a WRF and let $\mathbf{g}$ be a function which witnesses $\mathbf{MEET}$ with respect to this WRF. Then $\mathbf{h}$ given as $\mathbf{h(x) = g(x) + 1}$ witnesses $\mathbf{HI}$ with respect to this WRF.

Assume that $\{\mathbf{f_e} : \mathbf{e} \in \mathbf{E}\}$ is a WRF and let $\mathbf{F_e(x)}$ be the time to compute $\mathbf{f_e}$ up to $\mathbf{x}$. Let $\mathbf{h}$ satisfy $\mathbf{HI}$ with respect to $\{\mathbf{F_e} : \mathbf{e} \in \mathbf{E}\}$. Let $(\mathbf{e}, \mathbf{k})$ be the pairs enumerated in order of the set $\mathbf{M}$ with $<$. For each input $\mathbf{x}$ find the first $(\mathbf{e}, \mathbf{k})$ such that $\mathbf{Cond(e, k)}$ holds: $\mathbf{F_e(x) < h(x)}$ and $\mathbf{f_e(y)}$ and $\mathbf{g(y)}$ coincide exactly for $\mathbf{k - 1}$ numbers $\mathbf{y < x}$. Then let $\mathbf{g(x) = f_e(x)}$. By $\mathbf{I\Sigma^0_1}$ and the choice of $\mathbf{h}$ there is a number $\mathbf{x}$ such that there are no pair $(\mathbf{e'}, \mathbf{k'})$ before $(\mathbf{e}, \mathbf{k})$ such that $\mathbf{Cond(e', k')}$ holds at any $\mathbf{x' \geq x}$ and thus thus $\mathbf{g(x) = f_e(x)}$ will eventually made true at the first $\mathbf{x' \geq x}$ satisfying $\mathbf{Cond(e, k)}$. So $\mathbf{HI}$ implies $\mathbf{MEET}$.

# Conservativeness

Definition
An axiom $\mathbf{AX}$ is restricted $\mathbf{\Pi^1_2}$-conservative over $\mathbf{BX}$ iff for every arithmetical formula $\alpha$ with one free set variable and every $\mathbf{\Sigma^0_3}$-formula $\beta$ with two free set variables all statements provable from $\mathbf{AX}$ of the form

$$\forall \mathbf{X}\,[\alpha(\mathbf{X}) \rightarrow \exists \mathbf{Y}\,[\beta(\mathbf{X}, \mathbf{Y})]]$$

are also provable from $\mathbf{BX}$.

Theorem [Referee of HRSZ 2017]. $\mathbf{DOM}$ is restricted $\mathbf{\Pi^1_2}$-conservative over $\mathbf{RCA_0}$.

Theorem [HRSZ 2017]
$\mathbf{DOM}$ is not $\mathbf{\Pi^1_1}$-conservative over $\mathbf{B\Sigma^0_2}$, as $\mathbf{B\Sigma^0_2} + \mathbf{DOM} \vdash \mathbf{I\Sigma^0_2}$.

# Inductive Inference

A WRF $\{A_e : e \in E\}$ is learnable (from positive data) iff there is a learner $L : M^* \to M$ which for every $f \in S$ whose range is an $A_e$ satisfies that there is a $d \in M$ such that $L(f(0) \ldots f(x)) = d$ for almost all $x$ and $A_d = A_e$.

Angluin characterised when uniformly recursive families are learnable from positive data — Translated to this setting, the tell-tale condition says: There is a function $TT$ such that, for all $d, e \in E$, if $A_e \cap \{0, 1, \ldots, TT(e)\} \subseteq A_d \subseteq A_e$ then $A_d = A_e$.

One says that Angluin's tell-tale condition is
(a) satisfied effectively iff $TT \in S$,
(b) satisfied in the limit iff $TT(e) = \lim_s TT_s(e)$ for an approximation in $S$ and
(c) satisfied non-effectively iff no constraints on $TT$ are given.

# Characterising DOM

Theorem [HJS 2016]. The following conditions are all equivalent to DOM.

(a) The index set of any WRF is limit-recursive;

(b) Angluin's limit-recursive tell-tale condition characterises learnability;

(c) Angluin's non-effective tell-tale condition characterises learnability;

(d) Angluin's effective tell-tale condition implies learnability.

Furthermore, DOM is equivalent to the statement that every uniformly represented family is learnable iff it satisfies Angluin's limit-recursive tell-tale condition.

# Conservative Learnability

A learner $\mathbf{L}$ is conservative iff whenever $\mathbf{f(0), f(1), \ldots, f(x), f(x+1) \in A_{L(f(0)\,f(1)\,\ldots\,f(x))}}$ then $\mathbf{L(f(0)\,f(1)\,\ldots\,f(x)\,f(x+1)) = L(f(0)\,f(1)\,\ldots\,f(x))}$.

Theorem [HJS 2016]. A uniformly represented family is conservatively learnable using a family $\mathbf{\{A_e : e \in M\}}$ as hypothesis space iff it is contained in a family $\mathbf{\{B_e : e \in M\}}$ which, in turn, satisfies Angluin's tell-tale criterion effectively.

Theorem [HJS 2016]. The following two conditions are equivalent to $\mathbf{ACA_0}$:

(a) Every uniformly represented family that satisfies the tell-tale criterion in general is conservatively learnable;

(b) Every weakly represented family that satisfies the tell-tale criterion in general is conservatively learnable.

# Partial Learning

Definition [Osherson, Stob and Weinstein 1986].
A leaner $\mathbf{L}$ partially learns a WRF $\{\mathbf{A_e} : \mathbf{e} \in \mathbf{E}\}$ iff for every $\mathbf{f} \in \mathbf{S}$ with $\mathbf{range(F)}$ in the WRF there is exactly one index $\mathbf{e}$ which appears infinitely often in the sequence of the $\mathbf{a_x} = \mathbf{L(f(0)\,f(1)\,\ldots\,f(x))}$ and this $\mathbf{e}$ satisfies $\mathbf{A_e} = \mathbf{range(f)}$.

Theorem [Osherson, Stob and Weinstein 1986].
The class of all r.e. sets is partially learnable.

Theorem [HJS 2016].
Over $\mathbf{I\Sigma^0_2}$, every weakly represented family is partially learnable using a class-preserving hypothesis space.

# Maximal Families

$\mathbf{MAD}$: There is a weakly represented family $\{\mathbf{A_e} : \mathbf{e} \in \mathbf{E}\}$ such that whenever $\mathbf{d} \neq \mathbf{e}$ then $\mathbf{A_d} \cap \mathbf{A_e}$ is finite and all $\mathbf{A_e}$ are infinite and every infinite $\mathbf{B} \in \mathbf{S}$ has an infinite intersection with some $\mathbf{A_e}$.

$\mathbf{MED}$: There is a weakly represented family $\{\mathbf{f_e} : \mathbf{e} \in \mathbf{E}\}$ such that whenever $\mathbf{d} \neq \mathbf{e}$ then $\mathbf{f_d}$ and $\mathbf{f_e}$ coincide only finintely often and every further function $\mathbf{g} \in \mathbf{S}$ coincides with some $\mathbf{f_e}$ infinitely often.

Let $\mathbf{A^{-1}}$ be the complement of $\mathbf{A}$ and $\mathbf{A^1} = \mathbf{A}$.

$\mathbf{MIND}$: There is a weakly represented family $\{\mathbf{A_e} : \mathbf{e} \in \mathbf{E}\}$ such that for every finite $\mathbf{D} \subseteq \mathbf{E}$ and every $\mathbf{a_d} \in \{-\mathbf{1}, \mathbf{1}\}$ for $\mathbf{d} \in \mathbf{D}$ the set $\bigcap_{\mathbf{d} \in \mathbf{D}} \mathbf{A_d^{a_d}}$ is infinite and every infinite $\mathbf{B} \in \mathbf{S}$ contains some set of the form $\bigcap_{\mathbf{d} \in \mathbf{D}} \mathbf{A_d^{a_d}}$ as a subset.

# Implications

Observation [HRSZ 2017, CHRS 2018]. $\mathbf{DOM}$ implies $\mathbf{Not(MAD)}$, $\mathbf{BI}$ implies $\mathbf{Not(MIND)}$, $\mathbf{AVOID}$ implies $\mathbf{Not(MED)}$.

Proof. Let $\{A_e : e \in E\}$ be a family of almost disjoint infinite sets and $E_s$ be a limit-approximation of $E$ in $S$. Now let $B = \{b_0, b_1, \ldots\}$ be defined such that $b_0 = 0$ and $b_{n+1}$ is the first element after $b_n$ found such that, for all $m \leq n$, either $\exists t \geq s\,[E_t(m) = 0]$ or $A_e(m) = 0$. Now $B$ is infinite and has with all $A_e$ only a finite intersection.

If $\mathbf{BI}$ is satisfied then there is a $B$ which does not contain any subset of the form $\bigcap_{d \in D} A_d^{a_d}$.

If $\{f_e : e \in E\}$ satisfies $\mathbf{AVOID}$ then there is a $g$ which is eventually different from all $f_e$ with $e \in E$ and thus the family is not maximal.

# Reverse Implications

Theorem [HRSZ 2017]. Over $\omega$-models (that is, models where $\mathbf{M} = \mathbb{N}$), $\mathbf{Not(DOM)}$ implies $\mathbf{MAD}$ and $\mathbf{Not(BI)}$ implies $\mathbf{MIND}$.

Theorem [CHRS 2018]. Over $\mathbf{I\Sigma^0_2}$, $\mathbf{Not(DOM)}$ implies $\mathbf{MAD}$ and $\mathbf{Not(AVOID)}$ implies $\mathbf{MED}$ and $\mathbf{Not(BI)}$ implies $\mathbf{MIND}$.

Theorem [CHRS 2018]. Over $\mathbf{B\Sigma^0_2} + \mathbf{Not(I\Sigma^0_2)}$, $\mathbf{Not(DOM)}$ and $\mathbf{MAD}$.

Theorem [CHRS 2018]. There are models of $\mathbf{Not(B\Sigma^0_2)}$ which satisfy and models which do not satisfy $\mathbf{MAD}$; Similarly, there are models of $\mathbf{Not(B\Sigma^0_2)}$ which satisfy and models which do not satisfy $\mathbf{MED}$.

# Proof for $\mathbf{B\Sigma^0_2 + Not(I\Sigma^0_2) \vdash MAD}$

Assume $\mathbf{B\Sigma^0_2}$ and $\mathbf{Not(I\Sigma^0_2)}$. Then there is an $\mathbf{\Pi^0_2}$ initial segment $\mathbf{I}$ which is closed under successor and not a $\mathbf{\Sigma^0_2}$ set. One can find a function $\mathbf{f \in S}$ such that, for all $\mathbf{a \notin I}$ and almost all $\mathbf{b \in M}$, $\mathbf{f(a) \leq b}$ while, for all $\mathbf{a \in I}$, $\mathbf{f(b) > a}$ for infinitely many $\mathbf{b}$. Without loss of generality, for each $\mathbf{a \in I}$ there are infinitely many $\mathbf{b}$ with $\mathbf{f(b) = a}$. Let $\mathbf{\{A_a = \{b : f(b) = a\} : a \in I\}}$ be the given family, all members are disjoint and thus almost disjoint. Let $\mathbf{B \in S}$ be infinite. If $\mathbf{A_a \cap B}$ is finite for all $\mathbf{a \in I}$ then $\mathbf{A_a \cap B}$ is finite for all $\mathbf{a \leq c}$ for any given upper bound $\mathbf{c}$ of $\mathbf{I}$; by $\mathbf{B\Sigma^0_2}$ there would be an upper bound for the union of all these finite sets and thus $\mathbf{B}$ would not have any elements beyond that bound, a contradiction. Thus the family is a maximal almost disjoint family.

Question: Does such a family also exist with the additional property that the index set is unbounded?

# Conclusion

Weakly represented families are a useful tool to formulate recursion-theoretic, learning-theoretic and notions from cardinal invariants in reverse mathematics.

The paper HRSZ 2017 laid the foundations of this research and established the basic connections between the various branches of the field.

The paper HJS 2016 investigated the learnability of weakly represented families and linked the axioms $\mathbf{DOM}$ and $\mathbf{ACA_0}$ to various learning-theoretic theorems.

The paper CHRS 2018 improves equivalences which were in HRSZ 2017 only shown for $\omega$-models to models satisfying $\mathbf{B\Sigma^0_2}$ ($\mathbf{MAD} \leftrightarrow \mathbf{Not(DOM)}$) or $\mathbf{I\Sigma^0_2}$ ($\mathbf{MED} \leftrightarrow \mathbf{Not(AVOID)}$, $\mathbf{MIND} \leftrightarrow \mathbf{NOT(BI)}$).