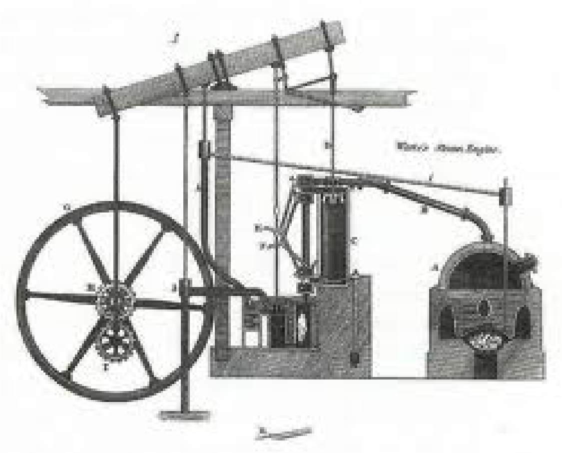

Strategies for Σ^0_3-games: an outline

• The determinacy of this class of games was proven by Morton Davis[1] and this is a proof in *analysis*.

For, recursively open, that is Σ^0_1, games the answer to the question *"Where do the strategies lie?"* are well known: they occur definably over $L_{\omega_1^{ck}}$ in the constructible hierarchy. Answers are known for Σ^0_2 as well.

• But where are the strategies Σ^0_3-games?

One can show that the strength of $Det(\Sigma^0_3)$ lies strictly between Π^1_3-CA_0 and Π^1_2-CA_0, and indeed we can pin down an exact level, L_{β_0} for this.[2]

• However today we focus on new work relating this exact level to notions generalising Kleene's generalised recursion theory of finite types, using Infinite Time Turing machines, rather than regular TM's.

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[1] M. Davis *"Infinite Games of Perfect Information"*, Ann. Math. Studies, 1964

[2] P.D. Welch *"Weak systems of analysis, determinacy and arithmetical quasi-inductive definitions"*, JSL, 2011

Kleene's Recursion in finite types

- $n \in \mathbb{N}$ are type 0; $x : \mathbb{N} \to \mathbb{N}$ are type 1; $\mathsf{F} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are type 2 ....

- Kleene:[3] gave a theory of *recursion in finite type objects* based on the Gödel-Herbrand type approach of an $\boxed{\textit{equational calculus.}}$

- Ordinary recursion: usual notion: $\{e\}(\boldsymbol{m}, \boldsymbol{x})\downarrow$

- A useful Type-2 functional: oJ - the *ordinary Turing jump functional*:

$$\mathsf{oJ}(e, \boldsymbol{m}, \boldsymbol{x}) = \begin{cases} 1 & \text{if } \{e\}(\boldsymbol{m}, \boldsymbol{x})\downarrow \\ 0 & \text{otherwise.} \end{cases}$$

[3] 1959 & 1963, Trans. of Amer. Math. Soc.

• This gives rise to a class of functions *recursive in* I for some type-2 functional I: in $\{e\}^{\mathsf{I}}$ an extra operation is allowed: consulting I.
There is the notion of one functional I_1 being recursive in another I_2.

• However now a recursion-in-I is best represented by a *well-founded but possibly infinitely branching tree.*

Theorem (Kleene)

(i) The oJ-recursive sets of integers, i.e. *those sets R for which*

$$R(n) \leftrightarrow \{e\}^{\mathsf{oJ}}(n)\downarrow 1 \ \wedge \ \neg R(n) \leftrightarrow \{e\}^{\mathsf{oJ}}(n)\downarrow 0$$

for some index e, are precisely the hyperarithmetic *ones.*

(ii)

$$H^{\mathsf{oJ}}(e) \leftrightarrow \{e\}^{\mathsf{oJ}}(e)\downarrow$$

is a complete semi-recursive (in oJ*) set of integers, and is a complete* Π^1_1 *set of integers:* $H^{\mathsf{oJ}} \equiv_1 \mathcal{O}$.

• In two further papers[4] he gave an equivalence to the *equational calculus version* of generalised recursion to one using *a* $\boxed{\textit{Turing machine model}}$.

[4] 1962: Proc. Lond. Math. Soc. & Proc. of CLMPS, Stanford)

$\Game\Sigma^0_n$ sets

Definition

For a universal Σ^0_n set $U \subseteq \mathbb{N} \times {}^{\mathbb{N}}\mathbb{N}$ then the set $\Game U$ which so arises is then a *complete* $\Game\Sigma^0_n$ *set*, and it essentially lists those Σ^0_n games that player I wins.

Definition

Let $\mathsf{G}^{\Sigma^0_1}$ denote the complete $\Game\Sigma^0_n$ set.

We have the following theorem which will connect this with determinacy of open games:

Theorem (Moschovakis, Svenonius)

The complete $\Game\Sigma^0_1$ set of integers, $\mathsf{G}^{\Sigma^0_1}$, is a complete Π^1_1 set of integers.

Further:

Theorem (Spector)

The ordinal of monotone Π^1_1 (and so $\Game\Sigma^0_1$) inductive definitions is ω_1^{ck}.

Moreover:

Theorem (Blass)

Any Σ^0_1-game for which the open player, that is I, has a winning strategy, has a HYP winning strategy. (Or in the above terms, an oJ*-recursive strategy.)*

Theorem (Summary)

$\mathsf{G}^{\Sigma^0_1} \equiv_1 H^{\mathsf{oJ}} \equiv_1 \mathcal{O} \equiv_1 T^1_{\omega_1^{ck}}$ *- the latter the* Σ_1*-Theory of* $(L_{\omega_1^{ck}}, \in)$.

We seek to raise all these ideas to the level of Σ^0_3.

• We first consider Kleene recursion in type 2 objects, but replacing Turing jump by the notion of *eventual jump* eJ derived from Hamkins' and Kidder's notion of an *infinite time Turing machine) (ITTM).*

• Lubarsky[5] already defined a related notion of *freezing-ittm-computations* which uses instead oracles for properly halting ittms arranged in well-founded trees. This kind of computation can also be formulated as a notion as here of recursion in a suitably defined *halting jump*, hJ.

[5] R. Lubarsky, *"Well founded iterations of Infinite Time Turing Machines"*, Ways of Proof Theory, Ed. R-D. Schindler, Ontos, 2010.

Part I: ITTM description[6]

• Allow a standard Turing machine to run transfinitely using one of the usual programs $\langle P_e \mid e \in \mathbb{N}\rangle$.
• Alphabet: $\{0, 1\}$;
• Enumerate the cells of the tape $\langle C_k \mid k \in \mathbb{N}\rangle$.
  Let the current instruction about to be performed at time τ be $I_{i(\tau)}$;
  Let the current cell being inspected be $C_{p(\tau)}$.
• Behaviour at successor stages $\alpha \to \alpha + 1$: as normal.
At limit times λ: (a) we specify by *fiat* cell values by:

$$C_k(\lambda) = \mathit{Liminf}_{\beta\to\lambda} C_k(\alpha)$$

(where the value in C_k at time τ is $C_k(\tau)$).
(b) we also (i) put the Read/Write head to cell $C_{p(\lambda)}$ where

$$p(\lambda) = \mathit{Liminf}^*_{\alpha\mapsto\lambda}\{p(\beta) \mid \alpha < \beta < \lambda\};$$

(ii) set

$$i(\lambda) = \mathit{Liminf}_{\alpha\mapsto\lambda}\{i(\beta) \mid \alpha < \beta < \lambda\}.$$

[6] Hamkins & Lewis *"Infinite Time Turing Machines"*, JSL, vol. 65, 2000.

• Hamkins & Lewis proved there is a *universal machine*, an S^m_n*-Theorem*, and a *Recursion Theorem* for ITTM's, and a wealth of results on the resulting ITTM-*degree theory.*

Definition (eventual, or settled output)

$P_e(x)\downarrow$ iff
There is a time β so that $P_e(x)$ has a fixed output tape for all later times α.

• We may define *eventual convergence* sets:

$$H = \{(e,x) \mid e \in \mathbb{N}, x \in 2^{\mathbb{N}} \wedge P_e(x)\downarrow\}$$

$$H_0 = \{e \mid e \in \mathbb{N} \wedge P_e(0)\downarrow\}$$

Q. What is H or H_0? (They are complete *ittm*-semi-decidable sets.)
How do we characterise them?
Q. How long do we have to wait to discover if $e \in H_0$ or not?

- Let $s(\alpha, e, x)$ be the *snapshot of the state of* $P_e(x)$ at time α.
- There is a cub set $D(e, x) \subseteq \omega_1$ s.t. $s(e, x, \alpha) = s(e, x, \beta)$.

The λ, ζ, Σ-Theorem[7]

Theorem

Let ζ be the least ordinal so that there exists $\Sigma > \zeta$ with the property that

$$L_\zeta \prec_{\Sigma_2} L_\Sigma; \qquad (\zeta \textit{ is } \text{“}\Sigma_2\text{-extendible”.)}$$

(i) The universal *ittm on integer input first enters a loop at time ζ.*
(ii) Then $\zeta = sup\{\alpha \mid \exists e\ P_e(0)\downarrow in\ \alpha\ steps\}$

By the Σ_2 nature of the ittm's, this means for any e, n,

$$s(\zeta, e, n) = s(\Sigma, e, x).$$

- As a corollary one derives a Normal Form Theorem and:

Corollary

$$H_0 \equiv_1 \Sigma_2\text{-}Th(L_\zeta).$$

[7] Welch *The length of ITTM computations*, Bull. London Math. Soc. 2000

Σ_n-nested ordinals

We say β admits a Σ_2-*nesting* if
(i) L_β is the well founded part of some model M of *KP* in which:
(ii) there are

$$\gamma_0 \leq \cdots \gamma_n \leq \cdots \quad \beta \quad \cdots < c_n < \cdots < c_0$$

with

$$(L_{\gamma_i} \prec_{\Sigma_2} L_{c_i})^{\mathcal{M}}.$$

Let β_0 be least that admits a Σ_2-nesting.

Theorem

Let δ be the least ordinal so that strategies for Σ^0_3 games are definable over L_δ. Then:

$$\delta = \beta_0.$$

We'd like a better characterisation of this ordinal than *via* nestings.

Functions generalised recursive in eJ

We generalise Kleene to a notion of type-2 recursion involving *ittm*'s rather than ordinary *tm*'s at nodes on a well founded tree. Our intention is that such machines may also make oracle calls concerning the eventual behaviour of other machines. We call this *ittm-generalised-recursion* (-in-eJ).

$$\mathsf{eJ} = \{\langle\langle f, \boldsymbol{m}, \boldsymbol{x}\rangle, i\rangle \ : \ (i = 1 \text{ and } P_f^{\mathsf{eJ}}(\boldsymbol{m}, \boldsymbol{x}) \text{ has fixed output}) \text{ or}$$
$$(i = 0 \text{ and } P_f^{\mathsf{eJ}}(\boldsymbol{m}, \boldsymbol{x}) \text{ does not have fixed output }) \}$$

Definition (The $\{e\}$'th function generalised recursive in eJ)

(i) $\{e\}^{\mathsf{eJ}}(\boldsymbol{m}, \boldsymbol{x})\downarrow$ iff $\langle e, \boldsymbol{m}, \boldsymbol{x}\rangle \in \mathrm{dom}(\mathsf{eJ})$

($\{e\}^{\mathsf{eJ}}(\boldsymbol{m}, \boldsymbol{x})$ is *defined* or *convergent*) ;

and $\{e\}^{\mathsf{eJ}}(\boldsymbol{m}, \boldsymbol{x}) = \mathsf{eJ}(e, \boldsymbol{m}, \boldsymbol{x})$.

(ii) Otherwise it is *undefined* or *divergent* ($\{e\}^{\mathsf{eJ}}(\boldsymbol{m}, \boldsymbol{x})\uparrow$).

Recall our summary concerning generalised recursion in oJ:

Theorem (Summary - Kleene Recursion in oJ)

$\mathsf{G}^{\Sigma^0_1} \equiv_1 H^{\mathsf{oJ}} \equiv_1 \mathcal{O} \equiv_1 T^1_{\omega_1^{ck}}$ *- the latter the Σ_1-Theory of $(L_{\omega_1^{ck}}, \in)$.*

Let $H^{\mathsf{eJ}} = \{e \mid \{e\}^{\mathsf{eJ}}(e)\downarrow\}$ be the complete eJ-semi-recursive set.

Theorem (Summary -generalised Kleene Recursion in eJ)

$$\mathsf{G}^{\Sigma^0_3} \equiv_1 H^{\mathsf{eJ}} \equiv_1 T^1_{\beta_0} \text{ - \textit{the latter the} } \Sigma_1\text{\textit{-Theory of} } (L_{\beta_0}, \in).$$

Generalising Blass

Recall:

Theorem (Blass)

Any Σ^0_1-game for which the open player, that is I, has a winning strategy, has a winning strategy in $L_{\omega_1^{ck}}$. (Or in the above terms, an oJ-recursive strategy.*)*

We have:

Theorem

Any Σ^0_3-game for which the open player, that is I, has a winning strategy, has a winning strategy in L_γ. (Or in the above terms, an eJ-recursive strategy.*)*

The length of monotone-$\Game\Sigma^0_3$-inductive operators

Theorem

If $\gamma < \beta_0$ is least with $L_\gamma \prec_{\Sigma_1} L_{\beta_0}$ then γ is the closure ordinal of monotone-$\Game\Sigma^0_3$-Inductive Operators.

More recently Hachtman has announced:

Theorem (Hachtman)

L_{β_0} is the least β-model of Π^1_2-MI.

So this gives yet another characterisation of β_0.