When a strong reduction is considered

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Reductions which are stronger

- Turing reduction
- m-reduction
- truth-table reduction
- weak-truth-table reduction, i.e. bT-reduction

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Degrees:

- Turing degrees, *wtt*-degrees, *tt*-degrees, *m*-degrees
- Computably enumerable *r*-degrees, Δ^0_2 *r*-egrees, all *r*-degrees
- Structural and Model-Theoretical properties

Computably Enumerable Turing-degrees

- Sacks splitting theorem
- Sacks density theorem
- Shoenfield conjecture
  - Lachlan-Yates minimal pair theorem: an instance of construction

  - There are nonrecursive c.e. sets such that

    If C is Turing reducible to both A and B, then C is recursive.

    Requirements:

    $$\mathcal{R}_e : \Phi_e^A = \Phi_e^B = f \text{ total } \Rightarrow f \text{ recursive.}$$

- Lattice embeddings/nonembeddings
- Lachlan's nonsplitting theorem and monster constructions

Computably Enumerable *wtt*-degrees

wtt-reduction was first proposed by Friedberg and Rogers, around 1956.

- Reconstider the requirement

$$\mathcal{R}_e : \Phi_e^A = \Phi_e^B = f \text{ total} \Rightarrow f \text{ recursive.}$$

  - Ladner and Sasso - Splitting+Density is true for the c.e. *wtt*-degrees.
  - Each c.e. Turing degree either contains one or infinitely many c.e. *wtt*-degrees.
  - Each nonzero c.e. *wtt*-degree contains a simple set.

Turing Jump and High/Low Hierarchy

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Jump Inversion Theorems:

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Jump Inversion Theorems:

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- Low degrees and High degrees, High/Low hierarchy
- Sacks: There exist intermediate c.e. degrees, i.e. not low_n, not high_n for any n.

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  - If A is superlow and $\emptyset'$ is *wtt*-reducible to $A \oplus W$, then $\emptyset'$ is *wtt*-reducible to W.

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  - If A is superlow and $\emptyset'$ is *wtt*-reducible to $A \oplus W$, then $\emptyset'$ is *wtt*-reducible to W.
- This shows the existence of c.e. sets, low, but not superlow.

Bounded Jump Operator - a definition of Anderson and Csima

For $A \subseteq \mathbb{N}$, define

$$A^\dagger = \{x : \exists i < x[\varphi_i(n) \downarrow \ \& \ \Phi_x^{A \upharpoonright \varphi_i(x)}(x) \downarrow]\}.$$

Obviously, $A^\dagger \leq_T A \oplus \emptyset'$, so if $A \geq_T \emptyset'$, then $A^\dagger \equiv_T A$.

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We can also have:

- $\emptyset^\dagger \equiv_1 \emptyset'$. So, for set A, $A \leq_{wtt} \emptyset^\dagger$ if and only if A is ω-c.e.
- $A \leq_1 A^\dagger$ and
- $A^\dagger \not\leq_{wtt} A$.
- $A^\dagger \leq_1 A'$.
- For some set A, $A^\dagger \not\leq_{wtt} A \oplus \emptyset'$.

  As indicated above, $A^\dagger \leq_T A \oplus \emptyset'$ is always true.
- For sets A, B with $A \leq_{wtt} B$, $A^\dagger \leq_1 B^\dagger$.

An analogue of Shoenfield's Jump Inversion

Theorem (Anderson and Csima):

For a set C with $\emptyset^\dagger \leq_{wtt} C \leq_{wtt} \emptyset^{\dagger\dagger}$, there is a set $B \leq_{wtt} \emptyset^\dagger$ such that $C \equiv_{wtt} B^\dagger$.

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Still open.

Bounded-low set

A set A is bounded-low if $A^{\dagger} \leq_{wtt} \emptyset^{\dagger}$, i.e. if $A^{\dagger}$ is ω-c.e..

1. All superlow sets are bounded low.
2. There is a high bounded-low set. (Anderson, Csima and Lange)
3. There is a superhigh bounded-low set. (Wu and Wu))
4. There is a low, but not superlow, bounded-low set. (Wu and Wu)
5. This provides answers to two questions of Anderson, Csima and Lange in their recent paper.

Thanks!