Trees with at most finitely many paths in reverse mathematics

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Trees with finitely many paths

We will consider the following well-known theorem.

Theorem

Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite computable tree. If T has at most finitely many paths, then T has a computable path.

Question

How can we understand this situation in reverse mathematics?

- "Any infinite tree $T \subseteq 2^{<\mathbb{N}}$ which has at most finitely-many paths has a path" is already equivalent to WKL since $\neg\mathrm{WKL}$ implies the existence of an infinite tree with no path.
- Thus, we will consider several structural conditions to support the finiteness of paths.

- A set $P \subseteq \mathbb{N}^{<\mathbb{N}}$ is said to be *prefix-free* if $\sigma \perp \tau$ for any $\sigma, \tau \in P$.

For a given tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, put

- $T^{=n} := \{\sigma \in T \mid |\sigma| = n\}$,
- $T_{\mathrm{ext}} := \{\sigma \in T \mid \forall n \in \mathbb{N}\, \exists \tau \in T^{=n}\, \tau \supseteq \sigma\}$.

We will consider the following versions of WKL.

1. WKL(*pf-bd*): an infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ has a path if there exists $c \in \mathbb{N}$ such that for any prefix-free set $P \subseteq T$, $|P| \leq c$.
2. WKL(*w-bd*): an infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ has a path if there exists $c \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $|T^{=n}| \leq c$, where $T^{=n} = \{\sigma \in T \mid \mathrm{lh}(\sigma) = n\}$.
3. WKL(*ext-bd*): an infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ has a path if there exists $c \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $|T^{=n}_{\mathrm{ext}}| \leq c$, where $T^{=n}_{\mathrm{ext}} = \{\sigma \in T \mid \mathrm{lh}(\sigma) = n \wedge \sigma$ is extendible$\}$.

* For a fixed standard $c \in \omega$, they are all provable within RCA_0.

We will also consider the following versions of KL.

1. $\mathrm{KL}(\textit{pf-bd})$: an infinite tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has a path if there exists $c \in \mathbb{N}$ such that for any prefix-free set $P \subseteq T$, $|P| \leq c$.
2. $\mathrm{KL}(\textit{w-bd})$: an infinite tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has a path if there exists $c \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $|T^{=n}| \leq c$.
3. $\mathrm{KL}(\textit{ext-bd})$: an finitely-branching infinite tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has a path if there exists $c \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $|T^{=n}_{\mathrm{ext}}| \leq c$.

For any Π^0_1 set $X \subseteq \mathbb{N}$, $\{X\} \subseteq \mathbb{N}^{\mathbb{N}}$ is a Π^0_1 singleton.
Thus, we have the following.

Proposition (RCA_0, folklore)

$\mathrm{KL}(\textit{ext-bd})$ *is equivalent to* ACA_0.

Induction strength

$\mathrm{WKL}(\textit{pf-bd})$, $\mathrm{WKL}(\textit{w-bd})$, $\mathrm{WKL}(\textit{ext-bd})$, $\mathrm{KL}(\textit{pf-bd})$ and $\mathrm{KL}(\textit{w-bd})$ are all true in ω-model of RCA_0.

Theorem

1. $\mathrm{WKL}(\textit{pf-bd})$ *is provable in* RCA_0*.*
2. $\mathrm{WKL}(\textit{w-bd})$ *and* $\mathrm{WKL}(\textit{ext-bd})$ *are provable in* $\mathsf{RCA}_0 + \mathrm{I}\Sigma^0_2$*.*
3. $\mathrm{KL}(\textit{pf-bd})$ *is provable in* $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$*.*
4. $\mathrm{KL}(\textit{w-bd})$ *is provable in* $\mathsf{RCA}_0 + \mathrm{I}\Sigma^0_2$*.*

Key idea: find a maximal prefix-free set $P \subseteq T$ so that any element of P has no “essential” branching.

1,3: take $b_0 = \max\{a \le b \mid \exists P \subseteq T(P \text{ is p-free and } |P| = a)\}$.

2: take $b_0 = \min\{a \le b \mid \forall n(|T^{=n}_{\mathrm{ext}}| \le a)\}$.

4: formalize Chaitin’s proof and combine with 2.

Converse

Theorem (RCA_0)

$\mathrm{KL}(\mathit{pf}\text{-}\mathit{bd})$ *is equivalent to* $\mathrm{B}\Sigma^0_2$.

Theorem (RCA_0)

$\mathrm{KL}(w\text{-}\mathit{bd})$ *is equivalent to* $\mathrm{I}\Sigma^0_2$.

The following lemma is essential for the second theorem.

Lemma (RCA_0)

If $\mathrm{I}\Sigma^0_2$ *fails, then there exists a set* X *and a* $\Pi^{0,X}_1$*-set* A *such that* A *is unbounded and* $|A| \leq c$ *for some* $c \in \mathbb{N}$.
(Here, A *is said to be unbounded if* $\forall n \in \mathbb{N} \exists m \geq n\ m \in A$*, and* $|A| \leq c$ *means that for any finite set* $F \subseteq A$ *(coded by a natural number),* $|F| \leq c$*.)*

Theorem (RCA_0)

$\mathrm{KL}(w\text{-}bd)$ *is equivalent to* $\mathrm{I}\Sigma^0_2$.

Proof.

$\neg\mathrm{I}\Sigma^0_2 \to \mathrm{KL}(w\text{-}bd)$: By the lemma, there exists a set X and a $\Pi^{0,X}_1$-set A such that A is unbounded and $|A| \leq c$ for some $c \in \mathbb{N}$. (Note that A cannot exist as a set.)
Write $n \in A \leftrightarrow \forall m\theta(m, n, X)$ where θ is a Σ^0_0-formula. Then, define a Σ^0_1 tree $T \subseteq 2^{<\mathbb{N}}$ as

$$\begin{aligned}\sigma \in T \leftrightarrow \exists m > \mathrm{lh}(\sigma)(\forall i < \mathrm{lh}(\sigma)(\sigma(i) = 1 \leftrightarrow \forall m' < m\,\theta(m', i, X))\\ \wedge |\{i < \mathrm{lh}(\sigma) \mid \forall m' < m\,\theta(m', i, X)\}| \leq c).\end{aligned}$$

Then, T is infinite and there are at most c-many elements in $T^{=n}$ for any $n \in \mathbb{N}$. A path of T should be identical with A, so T cannot have a path. $\square$

Is induction essentially needed?

Proposition (RCA_0)

WKL(*w-bd*) *and* WKL(*ext-bd*) *are equivalent.*

- Do they also require induction?

$\Rightarrow$ No! WKL(*ext-bd*) is provable from WKL_0 which is a Π^1_1-conservative extension of $\mathrm{I}\Sigma^0_1$.

Definition (Very smallness, Binns/Kjos-Hanssen)

VSMALL asserts the following: an infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ has a path if for any function $f : \mathbb{N} \to \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $|T^{=f(n)}_{\mathrm{ext}}| < n$.

- VSMALL is a very weak fragment of WKL which cannot imply DNR, but it is not derived from WWKL_0.

Is induction essentially needed?

Proposition (RCA_0)

WKL(*ext-bd*) *is provable from* $\mathrm{I}\Sigma^0_2 \vee \mathrm{VSMALL}$.

Thus, it is much weaker than WKL_0. However,

Theorem

WWKL_0 *does not imply* WKL(*ext-bd*).

The proof is very similar to the separation $\mathrm{WWKL}_0 \nRightarrow \mathrm{VSMALL}$ by Binns/Kjos-Hanssen.

Lemma (RCA_0, Simpson)

If X is a c.e. set such that $X >_T \mathbf{0}$, then X can be split into two c.e. sets $X = Y_0 \sqcup Y_1$ such that Y_0 and Y_1 are computably inseparable.

We will construct a model $(M, S) \models \mathrm{WWKL}_0 + \neg \mathrm{I}\Sigma^0_2$ but $\mathrm{WKL}(\textit{ext-bd})$ fails in it.

- Let $M \models \neg \mathrm{I}\Sigma^0_2$. Then, there exists a Π^0_1-set A such that A is unbounded and $|A| \leq c$ for some $c \in M$.
- Thus, in M, $A^c >_T \mathbf{0}$.
- Take c.e. sets in M $B_0 \sqcup B_1 = A^c$ such that $\mathrm{Sep}(B_0, B_1)$ has no computable member, and take a computable $T \subseteq 2^{<M}$ such that $[T] = \mathrm{Sep}(B_0, B_1)$.
- By a usual Harrington's forcing argument, there exists $Z \in \mathrm{MLR}$ such that $Z \not\geq_w [T]$.
- Then, there exists $S \leq_w Z$ such that $(M, S) \models \mathrm{WWKL}_0$.
- Since $T^{=n}_{\mathrm{ext}} \leq |A| \leq c$ and T has no path in S, $\mathrm{WKL}(\textit{ext-bd})$ fails in (M, S).

Is induction essentially needed?

Is induction essential for $\mathrm{WKL}(\textit{ext-bd})$?
$\Rightarrow$ Yes, in some sense.

Theorem (RCA_0)

$\mathrm{WKL}(\textit{ext-bd})$ *plus* $\exists X \forall Y (Y \leq_T X)$ *implies* $\mathrm{I}\Sigma^0_2$.

- Assume $\exists X \forall Y (Y \leq_T X)$ and $\neg \mathrm{I}\Sigma^0_2$. Then, there exists a $\Pi^{0,X}_1$-set A such that A is unbounded and $|A| \leq c$ for some $c \in \mathbb{N}$.
- Thus $A^c >_T X$.
- Take $\Sigma^{0,X}_1$-sets $B_0 \sqcup B_1 = A^c$ such that $\mathrm{Sep}(B_0, B_1)$ has no X-computable member, and take an X-computable $T \subseteq 2^{<\mathbb{N}}$ such that $[T] = \mathrm{Sep}(B_0, B_1)$.
- Since $T^{=n}_{\mathrm{ext}} \leq |A| \leq c$ and T has no path, $\mathrm{WKL}(\textit{ext-bd})$ fails.

Is induction essentially needed?

Question

- Does $\mathrm{WKL}(\textit{ext-bd})$ imply $\mathrm{I}\Sigma^0_2 \vee \mathrm{VSMALL}$?
- In general, does $\mathrm{WKL}(\textit{ext-bd})$ imply some weak fragment of WKL which is computably false in the absence of $\mathrm{I}\Sigma^0_2$?
  $\Rightarrow$ Yes, in a weak sense. It implies, at least, $\forall X \exists Y (Y \not\leq_T X)$.
  (Thus, we have $\mathrm{WKL}(\textit{ext-bd}) \Rightarrow \mathrm{I}\Sigma^0_2 \vee \forall X \exists Y (Y \not\leq_T X)$.)
  Does it imply something stronger?

Thank you!

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