Cut-Elimination Theorem for Π^1_1-CA

Ryota Akiyoshi

Faculty of Letters
Kyoto University

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Introduction

- The aim of this talk: to explain some basic ideas of proof theory (esp. cut-elimination theorem) for a “strong system” called Π^1_1-CA.

  esp., the collapsing theorem (impredicative cut-elimination).

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- Some history of proof theory (ordinal analysis):
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- Some history of proof theory (ordinal analysis):
  - The birth of proof theory (consistency proof) by D. Hilbert and W. Ackermann (ε-substitution).
  - The founder of ordinal analysis via cut-elimination: G. Gentzen (1936, 1938, 1943).
  - Gentzen's result: ε_0 is the least ordinal for showing the consistency of PA.
  - Ordinals after Gentzen: a tool for a classification of formal theories according to the "strength of a given theory" (proof-theoretic ordinal).
- Informally: the proof-theoretic ordinal of T = l.u.b. of the sizes of cut-free proofs in T or T^∞ (of some formula).

$$|PA| = \varepsilon_0, |RA| = \Gamma_0, |ID_1| = \psi(\varepsilon_{\Omega+1}) \dots$$

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- Some history of proof theory (after Gentzen):
  - Ordinal analysis for impredicative subsystems of analysis like Π^1_1-CA: G. Takeuti and his school (1950'-1970's).
  - German School (“infinitary proof theory”) : Schütte, Buchholz, Pohlers, Jäger (1970's-1980's).
  - The recent breakthrough of ordinal analysis up to Π^1_2-CA: Rathjen and Arai (1990's-).

Introduction

Another aspect of proof theory:

- The goal of Hilbert's program: to give finitistic meaning to ideal elements like $\forall, \exists$ in arithmetic with a restricted induction (ε-substitution).

- The main goal of Gentzen's consistency proof in 1936 is to give finitistic meaning to transfinite propositions of PA.

  cf. Dialectica interpretation, proof mining.

- Takeuti's work: a reduction of impredicative comprehension to inductive definition (ordinal diagrams).

- Works by German school: more transparent ways of reductions of impredicative comprehension to some constructive grounds (e.g., inductive definitions): Analysis of Impredicativity

Introduction

- Finitary proof theory by Gentzen-Takeuti-Arai and infinitary proof theory by Schütte-Buchholz-Pohlers-Rathjen.
- Advantages of infinitary proof theory:
  1. it is easy to understand cut-elimination theorems,
  2. ordinal notations are “read off” from cut-elimination procedures.
- In this talk we explain a major method: the Ω-rule by Buchholz.
- The Ω-rule: a reduction of Π^1_1-CA to inductive definitions.
- Another method due to Pohlers: Local predicativity.

cf. Thierry Coquand’s slides:
http://www.cse.chalmers.se/ coquand/proof.html

Part I: the Method of the ω-Rule.

Part II: the Method of the Ω-Rule.

Part III: An Extension of the Ω-Rule.

Part I: the Method of the ω-Rule.

Idea of the ω-Rule

- A germ of infinitary proof figure: Brouwer's proof for the Bar Induction(1927).
- Idea of infinitary proof figure: we can explicitly write all calculations or computations behind finite proof figure.

$$\dfrac{\begin{array}{c}\vdots\\ A(0)\end{array}\qquad\begin{array}{c}[A(x)]\\ \vdots\\ A(sx)\end{array}}{\forall xA}\ \text{Ind}
\qquad\qquad
\dfrac{\begin{array}{c}\vdots\\ A(0)\end{array}\quad\begin{array}{c}\vdots\\ A(0)\\ \vdots\\ A(s0)\end{array}\quad\begin{array}{c}\vdots\\ A(0)\\ \vdots\\ A(s0)\\ \vdots\\ A(ss0)\ldots\end{array}}{\forall xA}\ \omega\text{-rule}$$

- One observation by Gentzen: cut-rule behind induction.
- Another (hidden) observation by Takeuti: cut-rule behind Π^1_1-CA.

Idea of the ω-Rule

- ω-arithmetic (PA^∞): obtained by replacing induction axiom by the following infinitary rule:

$$\frac{A(0), A(1), \ldots \text{for all } n \in \omega}{\forall x A(x)}$$

- The index set of the ω-rule: the set of natural numbers.
- The ω-rule satisfies the subformula property.
- The full cut-elimination theorem will hold for PA^∞ while Gentzen proved a partial cut-elimination theorem for PA in 1938.

G. Gentzen, “Die Widerspruchfreiheit der reinen Zahlentheorie”, 1936 (implicit use of the ω-rule).

G. Gentzen, “Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie”, 1938 (c.e.for empty sequent).

Infinitary System PA^∞

Definition

Let L be the language for first-order arithmetic containing $\bigwedge, \bigvee, \forall, \exists$.

- L: without free number variables.
- Negation is defined by de Morgan's laws: $\neg(A \wedge B) := \neg A \vee \neg B$, $\neg(A \vee B) := \neg A \wedge \neg B$, $\neg \forall x A : \exists x \neg A$, $\neg \exists x A := \forall x \neg A$.

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- We use Tait's calculus (one-sided sequent calculus).
- Γ (or $\Delta, \dots$) : a set of formulas.
- Only principal and minor formulas are explicitly shown, and weakning and contraction are implicitly assumed.
- Formula A or $\neg A$ where A is atomic is called a *literal*, $\mathrm{TRUE} :=$ the set of all true literals (Ex: $2+1=3$).

Infinitary System PA^∞

Definition

Inference rules of PA^∞ :

$$(\mathrm{Ax}_\Delta)\ \Delta \text{ where } \Delta = \{A\} \subseteq \text{ TRUE or } \ \Delta = \{C, \neg C\}$$

$$(\textstyle\bigwedge_{A_0 \wedge A_1})\ \frac{A_0 \quad A_1}{A_0 \wedge A_1} \qquad (\textstyle\bigvee^k_{A_0 \wedge A_1})\ \frac{A_k}{A_0 \vee A_1} \text{ where } k \in \{0,1\}$$

$$(\textstyle\bigwedge_{\forall xA})\ \frac{\ldots A(x/n) \ldots \text{ for all } n \in \omega}{\forall xA} \qquad (\textstyle\bigvee^k_{\exists xA})\ \frac{A(x/k)}{\exists xA} \text{ where } k \in \omega$$

$$(Cut_A)\ \frac{A \quad \neg A}{\emptyset}$$

Cut-Elimination for PA^∞

Definition

$rk(A)$ is defined as follows.

1. $rk(A) := 0$ if A is a literal.
2. $rk(A \wedge B) := rk(A \vee B) := max(rk(A), rk(B)) + 1$.
3. $rk(\forall x A(x)) := rk(\exists x A(x)) := rk(A(0)) + 1$.

Definition

Let I be an inference symbol and $d \in PA^\infty$. Then $dg(I)$ and $dg(d)$ are defined as follows.

1. $dg(I) := rk(C) + 1$ if $I = Cut_C$.
2. $dg(I) := 0$ otherwise.
3. $dg(I(d_\tau)_{\tau \in I}) := sup(\{dg(I)\} \cup \{dg(d_\tau) | \tau \in I\})$.

Cut-Elimination for PA^{∞}

The notation $d \vdash_m \Gamma$: d is a derivation of Γ and its cut-rank is $\leq m$.

Theorem (One-Step Reduction)

We define an operator $\mathscr{R}_C$ such that
if $d_0 \vdash_m \Gamma, C$, $d_1 \vdash_m \Gamma, \neg C$ and $rk(C) \leq m$, then $\mathscr{R}_C(d_0, d_1) \vdash_m \Gamma$.

Proof. By induction on d_0 and d_1. Consider only the crucial cases:

Case 1.

If $d_0 = \mathrm{Ax}_{C,\neg C}$, then $d_0 \vdash_m \Gamma', \neg C, C$. Thus $\neg C \in \Gamma$. We define

$$\mathscr{R}_C(d_0, d_1) := d_1 \vdash_m \Gamma.$$

Case 2.

Assume that $d_0 = \bigwedge_{\forall xA}(d_{0i})_{i\in\omega}$ and $d_1 = \bigvee^{k}_{\exists xA}(d_{10})$ so that
$d_{0i} \vdash_m \Gamma, A(i), \forall xA$ and $d_{10} \vdash_m \Gamma, A(k), \exists xA$.

Cut-Elimination for PA^∞

By IH, we have $\mathscr{R}_C(d_{0k}, d_1) \vdash_m \Gamma, A(k)$ and $\mathscr{R}_C(d_0, d_{10}) \vdash_m \Gamma, \neg A(k)$.
Thus we obtain the required derivation by applying a new cut with its cut-formula $A(k)$.

$$\mathscr{R}_C(d_0, d_1) := Cut_{A(k)}(\mathscr{R}_C(d_{0k}, d_1), \mathscr{R}_C(d_0, d_{10})) \vdash_m \Gamma.$$

Note that $rk(A(k)) < rk(\forall x A) < m$. □

Remark: the subformula property $rk(A(k)) < rk(\forall x A)$.

Cut-Elimination for PA^∞

Then we can define an operator $\mathscr{E}$ reducing cut-rank by 1.

Theorem (Cut-Reduction)

We can define an operator $\mathscr{E}$ on derivations in PA^∞ such that if $d \vdash_{m+1} \Gamma$, then $\mathscr{E}(d) \vdash_m \Gamma$.

Proof. Let d be $Cut_C(d_0, d_1)$. Then $\mathscr{E}(d)$ is defined using $\mathscr{R}$.

$$\mathscr{E}(d) := \mathscr{R}_C(\mathscr{E}(d_0), \mathscr{E}(d_1)). \square$$

Moreover we can eliminate all cuts if we want.

Theorem (Predicative Cut-Elimination)

We can define an operator $\mathscr{E}_\omega$ on derivations in PA^∞ such that if $d \vdash_\omega \Gamma$, then $\mathscr{E}_\omega(d) \vdash_0 \Gamma$.

Proof. By induction on d. Note that $\mathscr{E}_\omega$ is defined by arbitrary finite applications of $\mathscr{E}$. $\square$

Cut-Elimination for PA^∞

Theorem (Embedding)

if $d \vdash \Gamma$ where $d \in PA$ and Γ: closed, then $d^\infty \vdash_m^{<\omega+\omega} \Gamma$ for some $m \in \omega$.

$$(\mathrm{Ind}^y_{F,t})\ \frac{\neg F(x), F(sx)}{\neg F(0), F(t)} \qquad (\bigwedge^y_{\forall xA})\ \frac{A(x/y)}{\forall xA}$$

- Moreover, if $d_0 \vdash_m^\alpha \Gamma, C$, $d_1 \vdash_m^\beta \Gamma, \neg C$ and $rk(C) \leq m$, then $\mathscr{R}_C(d_0, d_1) \vdash_m^{\alpha\sharp\beta} \Gamma$.
- So, if $d \vdash_{m+1}^\alpha$, then $\mathscr{E}(d) \vdash_m^{\omega^\alpha} \Gamma$.
- Thus we can understand why Gentzen used the principle of transfinite induction up to ε_0 because

$$\varepsilon_0 = sup\{\omega_n(\omega+\omega) | n \in \omega\}$$

where $\omega_0(\alpha) := \alpha$ and $\omega_{m+1}(\alpha) = \omega^{\omega_m(\alpha)}$.

Part II: the Method of the Ω-Rule.

Introduction

Definition

The language of BI (parameter-free) : if $\forall X A(X), \exists X A(X)$ are formulas, then $A(X)$ contains no second-order quantifier (arithmetical) and no free predicate variable other than X.

Definition

The inference rules of BI: ones for first-order logical connectives, arithmetical axiom, Cut, Schütte's ω-rule and the rules for second-order quantifiers：

$$\frac{A(Y)}{\forall X A(X)} \ \bigwedge_{\forall X A(X)} \qquad \frac{\neg A(T)}{\exists X \neg A(X)} \ \bigvee^{T}_{\exists X \neg A(X)}$$

- $\bigvee^{T}_{\exists X \neg A(X)}$ is just a parameter-free Π^1_1-CA.
- There is no subformula property in $\bigvee^{T}_{\exists X \neg A(X)}$($T$ can be complicated).

- In the case of impredicative theory, cut-elimination is very difficult.
- The typical derivation d with an impredicative cut where Γ is arithmetical:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma, A(X), \forall X A(X)\end{matrix}}{\Gamma, \forall X A(X)}\ \bigwedge_{\forall X A(X)} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma, \neg A(T), \exists X \neg A(X)\end{matrix}}{\Gamma, \exists X \neg A(X)}\ \bigvee^{T}_{\exists X \neg A(X)}}{\Gamma}\ Cut$$

- Takeuti transforms d into $red(d)$ by replacing Π^1_1-CA by the substitution rule Sub^X_T and new Cut:

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, A(X), \forall X A(X)\end{array}}{\Gamma, \forall X A(X)} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, A(X), \forall X A(X)\end{array} \quad \begin{array}{c}\vdots\\ \Gamma, \exists X \neg A(X)\end{array}}{\Gamma, A(X)}\,Cut}{\Gamma, A(T)}\,Sub^X_T \qquad \begin{array}{c}\vdots\\ \Gamma, \neg A(T), \exists X \neg A(X)\end{array}}{\Gamma, \exists X \neg A(X)}\,Cut}{\Gamma}\,Cut
$$

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  $\Longrightarrow$ No : new cuts are inserted in $red(d)$.
- Obvious idea 2 : the length (or height) of derivation as a tree

  $\Longrightarrow$ No : the length of $red(d)$ seems to be longer than d.

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  $\Longrightarrow$ The ordinal diagram assigned to $red(d)$ is smaller (in some sense) than one assigned to d.
- But the system of ordinal diagrams is very complicated (esp., relations between ordinal diagrams).
- It is quite difficult to understand how it works.

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  $\Longrightarrow$ The ordinal diagram assigned to $red(d)$ is smaller (in some sense) than one assigned to d.
- But the system of ordinal diagrams is very complicated (esp., relations between ordinal diagrams).
- It is quite difficult to understand how it works.
- Question: How to show the termination of a reduction step ???

Introduction

- Buchholz's Ω-rule: an intuitively easy way of proving c.e. for BI.

  Cf. Buchholz, "Explaining the Gentzen-Takeuti reduction steps", AML, 2001.
- Moreover: Buchholz's proof = Takeuti's one.

Development of the Ω-Rule

Basic results via the Ω-rule and its extensions:

- The birth of the Ω-rule: Buchholz's Habilitationsschrift, *Eine Erweiterung der Schnitteliminationsmethode*, 1977.
- Ordinal analysis for iterated inductive definitions: Buchholz, LNM 897, 1981.
- Ordinal analysis for subsystems of second-order arithmetic: Buchholz and Schütte, 1988.
- Game theoretic extension of the Ω-rule: Towsner, 2009.
- Complete cut-elimination theorem: Akiyoshi and Mints, 2011.

Development of the Ω-Rule

Other applications of the Ω-rule:

- Ordinal analysis for Krsukal's theorem: Rathjen and Weiermann, 1993.
- Partial cut-elimination for modal μ-calculus: Jäger and Studer, 2011.
- Complete cut-elimination for modal μ-calculus: Mints, 2012.

Advantages of the Ω-rule:

- the c.e. process is intuitively easy to understand,
- the formulation can be "ordinal-free",
- it gives a direct analysis of comprehension, especially, up to iteration of Π^1_1-CA.

Idea of the Ω-rule

- The Idea of $\Omega_{\neg\forall XA(X)}$:
  1. $\Gamma, \neg\forall XA$ is equivalent to $\forall XA \to \Gamma$.
  2. Proof of $\forall XA \to \Gamma$: a function from any proof of $\forall XA$ into a proof of Γ (BHK-reading).
  3. Thus, when we have a proof of Γ, Δ for any cut-free proof of $\forall XA, \Delta$, we can assert $\Gamma, \neg\forall XA$.

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  3. Thus, when we have a proof of Γ, Δ for any cut-free proof of $\forall X A, \Delta$, we can assert $\Gamma, \neg\forall X A$.
- Let q :

$$\begin{array}{c} \vdots \\ \Delta, A(X) \end{array}$$

  be an arithmetical cut-free proof of the sequent $\Delta, A(X)$ where Δ is arithmetical and $X \notin FV(\Delta)$.
- BI^{Ω}: an infinitary system with the $\Omega, \widetilde{\Omega}$-rules.

Formulation of the Ω-rule

Informal pictures of the $\Omega, \widetilde{\Omega}$-rules:

$$
(\Omega)\ \frac{\left\{ \begin{array}{c} \vdots \\ q : \Delta, A(X) \end{array} \right\} \quad \begin{array}{c} \vdots \\ \ldots \Gamma, \Delta, \neg\forall X A \ldots \end{array}}{\Gamma, \neg\forall X A}
\qquad
(\widetilde{\Omega})\ \frac{\Gamma, A(X) \qquad \left\{ \begin{array}{c} \vdots \\ q : \Delta, A(X) \end{array} \right\} \quad \begin{array}{c} \vdots \\ \ldots \Gamma, \Delta \ldots \end{array}}{\Gamma}\ !X!
$$

- The Ω-rule is used to interpret Π^1_1-CA.
- The index set $|\Omega|$ of Ω: the set of the derivations q.
- $\widetilde{\Omega}_{\neg\forall XA}$ is a combination of $\Omega_{\neg\forall XA(X)}, \bigwedge_{\forall XA}, Cut_{\forall XA(X)}$: hidden impredicative cut.

Formulation of the Ω-rule

Definition

Let BI_0^Ω be a cut-free ω-arithmetic. Define $|\Omega| :=$ the set of $q =< d, X >$ with $\mathrm{BI}_0^\Omega \ni d \vdash A(X), \Delta_q$ where Δ_q is arithmetical and X is a second-order variable.

$$\frac{\ldots A(n) \ldots}{\forall x A(x)} \bigwedge_{\forall x A(x)}$$

Definition

The inference rules of BI^Ω: ones for first-order connectives, arithmetical axioms, $\Omega, \widetilde{\Omega}$ and the following rules:

$$\frac{A \quad \neg A}{\emptyset} \; Cut \quad \frac{A(Y)}{\forall X A(X)} \bigwedge_{\forall X A(X)}$$

Formulation of the Ω-rule

The $\Omega, \widetilde{\Omega}$-rules:

$$(\Omega_{\neg\forall XA})\ \frac{\ldots\Delta_q\ldots\ \ q \in |\Omega|}{\neg\forall XA} \quad (\widetilde{\Omega}_{\neg\forall XA})\ \frac{A(X) \quad \ldots\Delta_q\ldots\ \ q \in |\Omega|}{\emptyset}\ !X!$$

- Since BI_0^Ω is ω-arithmetic, the set of $q =< d, X >$ is well-defined.
- By quantification over the set $|\Omega|$ of q, the $\Omega, \widetilde{\Omega}$-rules are defined.
- The definitions proceed by inductive definition.

Interpretation of Π^1_1-CA

- $rk(C)$: the usual logical complexity of a formula C except that $rk(\forall X A(X)) = rk(\exists X A(X)) = 0$.
- $dg(d)$: the cut-rank of d defined using $rk(C)$ where C is a cut-formula in it.
- We write $\vdash_m \Gamma$ if there is a derivation d such that its end-sequent is Γ and $dg(d) \leq m$.
- We define a one-step reduction operator $\mathscr{R}_C$ in BI^Ω as before.

Theorem (One-Step Reduction)

There is an operator $\mathscr{R}_C$ such that
If $d_0 \vdash_m \Gamma, C$, $d_1 \vdash_m \Gamma, \neg C$ and $rk(C) \leq m$, then $\mathscr{R}_C(d_0, d_1) \vdash_m \Gamma$.

Proof. By double induction on d_0 and d_1. $\mathscr{R}_C$ replaces impredicative cut by "hidden" impredicative cut $\widetilde{\Omega}$ and eliminates other cuts in the standard way.

Interpretation of Π^1_1-CA

Assume $d_0 = \bigwedge_{\forall XA}(d_{00}) \vdash_m \Gamma, \forall XA$, $d_1 = \Omega(\ldots d_{1q} \ldots) \vdash_m \Gamma, \neg\forall XA$. Then $d_{00} \vdash_m \Gamma, A(X), \forall XA$ and $d_{1q} \vdash_m \Gamma, \Delta, \neg\forall XA$.

Using IH twice, we get $\mathscr{R}_C(d_{00}, d_1) \vdash_m \Gamma, A(X)$ and $\mathscr{R}_C(d_0, d_{1q}) \vdash_m \Gamma, \Delta$. Hence we have the required derivation via $\widetilde{\Omega}$:

$$\widetilde{\Omega}(\mathscr{R}_C(d_{00}, d_1), \ldots \mathscr{R}_C(d_0, d_{1q}) \ldots) \vdash_m \Gamma. \square$$

Theorem (Substitution)

There is an operator $\mathscr{S}^X_T$ such that if $BI^\Omega_0 \ni d \vdash_0 \Gamma$, then $BI^\Omega_0 \ni \mathscr{S}^X_T(d) \vdash_0 \Gamma[X/T]$.

Proof. By induction on d. Notice that the operator preserves all inference rules in d. The last inference rule of d cannot be $Cut, \Omega, \widetilde{\Omega}$. $\square$

Remark: $\mathscr{S}^X_T$ is a key of the Ω-rule. In the iterated case, some care is needed.

Interpretation of Π^1_1-CA

- Consider an application of Π^1_1-CA in BI:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, \neg A(T), \neg\forall X A(X)\end{array}}{\Gamma, \neg\forall X A(X)}$$

- This derivation is embedded into BI^Ω in the following way:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Delta, A(X)\end{array}}{\Delta, A(T)}\ \mathscr{S}^X_T \qquad \begin{array}{c}\vdots\\ \Gamma, \neg A(T), \neg\forall X A\end{array}}{\ldots\Gamma, \Delta, \neg\forall X A\ldots}\ \mathscr{R}_{A(T)}}{\Gamma, \neg\forall X A}\ \Omega$$

Interpretation of Impredicative Cut

- How about the following impredicative cut ?

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma, A(X), \forall X A(X)\end{matrix}}{\Gamma, \forall X A(X)} \bigwedge_{\forall X A(X)} \quad \dfrac{\begin{matrix}\vdots \\ \Gamma, \neg A(T), \exists X \neg A(X)\end{matrix}}{\Gamma, \exists X \neg A(X)} \bigvee^{T}_{\exists X \neg A(X)}}{\Gamma} \; Cut$$

- Π^1_1-CA $\bigvee^{T}_{\exists X \neg A(X)}$ is interpreted by Ω.
- *Cut* is interpreted by $\mathscr{R}$.
- Thus the whole derivation is interpreted by $\widetilde{\Omega}$.

Interpretation of Impredicative Cut

$$
\dfrac{
\dfrac{\overset{\vdots}{\Gamma, A(X), \forall X A(X)} \quad \overset{\vdots}{\Gamma, \exists X \neg A(X)}}{\Gamma, A(X)}\,\color{red}{\mathscr{R}}
\qquad
\dfrac{\overset{\vdots}{\Gamma, \forall X A(X)} \quad \dfrac{\dfrac{\overset{\vdots}{\Delta, A(X)}}{\Delta, A(T)}\,\mathscr{S}^X_T \quad \overset{\vdots}{\Gamma, \neg A(T), \exists X \neg A(X)}}{\Gamma, \Delta, \exists X \neg A(X)}\,\color{red}{\mathscr{R}}}{\ldots \Gamma, \Delta, \ldots}\,\color{red}{\mathscr{R}}
}{\Gamma}\,\widetilde{\Omega}
$$

- The notation $\ldots\Gamma,\Delta\ldots$ means that we have a proof Γ,Δ for any given cut-free proof of $\Delta, A(X)$.

Interpretation of Impredicative Cut

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma, A(X), \forall X A(X)\end{matrix} \quad \begin{matrix}\vdots\\ \Gamma, \exists X \neg A(X)\end{matrix}}{\Gamma, A(X)}\,\mathscr{R} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma, \forall X A(X)\end{matrix} \quad \dfrac{\dfrac{\begin{matrix}\vdots\\ \Delta, A(X)\end{matrix}}{\Delta, A(T)}\,\mathscr{S}^X_T \quad \begin{matrix}\vdots\\ \Gamma, \neg A(T), \exists X \neg A(X)\end{matrix}}{\Gamma, \Delta, \exists X \neg A(X)}\,\mathscr{R}}{\ldots \Gamma, \Delta, \ldots}\,\mathscr{R}}{\Gamma}\,\widetilde{\Omega}$$

- The notation $\ldots\Gamma,\Delta\ldots$ means that we have a proof Γ,Δ for any given cut-free proof of $\Delta, A(X)$.
- Informal picture of collapsing (impredicative c.e.) : taking a subtree on the right hand side.
- Moreover: Takeuti's reduction = Buchholz's collapsing.

Predicative Cut-Elimination and Collapsing Theorem

- The operation $\mathscr{E}$ reducing cut-rank by 1 is defined as before:

Theorem (Cut-Reduction)

If $d \vdash_{m+1} \Gamma$, then $\mathscr{E}(d) \vdash_m \Gamma$.

Proof. $\mathscr{E}(Cut_C(d_0, d_1)) := \mathscr{R}_C(\mathscr{E}(d_0), \mathscr{E}(d_1))$. □

- Moreover we define the operator $\mathscr{D}$ eliminating hidden cut $\widetilde{\Omega}$ as Buchholz did. Recall that BI_0^Ω is cut-free ω-arithmetic.

Theorem (Collapsing)

If $BI^\Omega \ni d \vdash_0 \Gamma$ and Γ is arithmetical, then $BI_0^\Omega \ni \mathscr{D}(d) \vdash \Gamma$.

Proof. Since Γ is arithmetical, the last rule of d cannot be $\Omega, \bigwedge_{\forall XA}$. Moreover, it cannot be Cut. Hence the principal case is that the last rule of $d = \widetilde{\Omega}$. In other cases we apply IH.

Predicative Cut-Elimination and Collapsing Theorem

Let $d = \widetilde{\Omega}(d_0, \ldots d_q \ldots)$ such that $d_0 \vdash \Gamma, A(X)$ and $d_q \vdash \Gamma, \Delta$. Since $\Gamma, A(X)$ is arithmetical, we have $\mathrm{BI}_0^{\Omega} \ni \mathscr{D}(d_0) \vdash \Gamma, A(X)$.

Now taking Γ as Δ in d_q, then $d_{\mathscr{D}(d_0)} \vdash \Gamma$. Since $d_{\mathscr{D}(d_0)}$ is a sub derivation of d, so we apply IH again: $\mathrm{BI}_0^{\Omega} \ni \mathscr{D}(d_{\mathscr{D}(d_0)}) \vdash \Gamma$. ◻

$$\left\{ q : \begin{array}{c} \vdots \\ \Delta, A(X) \end{array} \right\}$$

$$(\widetilde{\Omega})\ \frac{\begin{array}{c}\vdots\\ \Gamma, A(X)\end{array} \qquad \begin{array}{c}\vdots\\ \ldots \Gamma, \Delta \ldots\end{array}}{\Gamma}\ !X!$$

A Simple Explanation of the Collapsing Theorem

Let $d = \widetilde{\Omega}(d_0, \ldots d_q \ldots)$ such that $d_0 \vdash \Gamma, A(X)$ and $d_q \vdash \Gamma, \Delta$.
Assume that there is the only one $\widetilde{\Omega}$ in d.

The key idea: d_0 can be regarded as an “input” q by taking Γ as Δ_q.

Then there is subderivation d_{d_0} of d, which is completely cut-free (without $\widetilde{\Omega}$). Moreover, $d_{d_0} \vdash \Gamma$ since $\Gamma, \Delta_{d_0} = \Gamma$.

The collapsing step is taking a subderivation of a given derivation. If needed, this process is iterated.

$$(\widetilde{\Omega})\ \dfrac{\begin{array}{c}\vdots\\ \Gamma, A(X)\end{array} \qquad \begin{array}{c}\left\{\begin{array}{c}\vdots\\ q : \Delta, A(X)\end{array}\right\}\\ \vdots\\ \ldots\Gamma,\Delta\ldots\end{array}}{\Gamma}\ !X!$$

Embedding Function from BI to BI^{Ω}

- The embedding function $()^{\infty}$ from BI to BI^{Ω} is a function such that :
  1. Π^1_1-CA $\Longrightarrow \Omega$
  2. impredicative cut $\Longrightarrow \widetilde{\Omega}$

Theorem (Embedding of BI into BI^{Ω})

If $BI \ni d \vdash \Gamma$, then $BI^{\Omega} \ni d^{\infty} \vdash \Gamma$.

Recall that BI^{Ω}_0 is cut-free ω-arithmetic. Combining these theorems obtained so far, we have the following:

Theorem

If $BI \ni d \vdash \Gamma$, then $BI^{\Omega}_0 \ni \mathscr{D}(\mathscr{E}^m(d^{\infty})) \vdash_0 \Gamma$ where Γ is arithmetical.

Part III: An Extension of the Ω-Rule.

Main Difficulty for Extending the Ω-Rule

- Complete cut-elimination: c.e. for arbitrary sequents.
- Why the collapsing is partial cut-elimination for arithmetical formulas?

  $\Rightarrow$ The domain of the Ω-rule contains only arithmetical proofs.
- The main difficulty to extend the Ω-rule is to define $|\Omega|$ (impredicativity of the Ω-rule).
- But: if $d \in |\Omega|$ ranges over (not only arithmetical) all proofs, we are led to circular.
- Reason: we cannot quantifier over a domain of the Ω-rule if it may contain the Ω-rule.

Main Difficulty for Extending the Ω-Rule

- The Ω-rule:
  1. Take any $q \in |\Omega|$ such that $q \vdash_0 A(x), \Delta$.
  2. If we have a proof $f(q) \vdash \Gamma, \neg\forall XA, \Delta$, then $\Omega(\ldots f(q) \ldots) \vdash \Gamma, \neg\forall XA$.
- The key: quantification over $|\Omega|$, the source of impredicativity of the Ω-rule (inductive definition).
- If $q \in |\Omega|$ might contain the Ω-rule, then we have assumed what we want to define, hence vicious circle.
- Our idea (joint with G.Mints): to extend the Ω-rule using Takeuti's notion of explicit/implicit inference.

Main idea

Main idea is to define the Ω-rule based on explicit/implicit distinction.

Definition (Takeuti)

A logical inference with the principal formula A is *implicit* if there is a cut whose cut-formula is traced to A. Otherwise it's called *explicit*.

- Example: any logical inference of a derivation of empty sequent is implicit (cf. Gentzen's consistency proof of PA).
- Observation: explicit/implicit distinction is preserved in the process of cut-elimination.

Main Idea

- The domain of the Ω^+-rule contains not only logical inferences for arithmetical formulas, but $\bigvee_{\neg\forall XB}$.
- Implicit Π^1_1-CA: translated into the Ω-rule as Buchholz did.
- Explicit Π^1_1-CA: preserved, hence included into $|\Omega^+|$.

Example:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, A(Y), \neg B(S)}}{\Gamma, A(Y), \neg\forall XB}\ (\bigvee_{\neg\forall XB})}{\Gamma, \forall XA, \neg\forall XB} \qquad \dfrac{\dfrac{\vdots}{\Gamma, \neg A(T)}}{\Gamma, \neg\forall XA}\ (\Omega)$$

$$\dfrac{\Gamma, \forall XA, \neg\forall XB \qquad \Gamma, \neg\forall XA}{\Gamma, \neg\forall XB}\ Cut(\widetilde{\Omega})$$

Formulation of Ω^+-Rule

Definition

Language of second-order arithmetic sorted by e, i.

Example: $X(t)^i, X(t)^e, \forall xA^e, \forall xA^i, \forall XA^e, \forall XA^e, \exists XA^i, \exists XA^e$.
Γ^e, Δ^e denotes sets of explicit formulas: Γ^e, Δ^e contains only formulas like A^e.
Sequent : Γ^e, Π^i.

Definition

$\mathrm{BI}_0^{\Omega^+}$: two sorted version of $\mathrm{BI}_0^{\Omega^+}$ with $\bigvee_{\neg\forall XA^e}$. Let $\tau, \iota \in \{e, i\}$.

Axiom: restricted to atomic C:

$$\frac{}{C^\tau, \neg C^\iota}\ \mathrm{Ax}_{C^\tau, \neg C^\iota}$$

$$!Y!\ \frac{A(Y)^\tau}{\forall XA^\tau}\ \bigwedge\nolimits_{\forall XA^\tau} \qquad \frac{\neg A(T)^e}{\neg\forall XA^e}\ \bigvee\nolimits_{\neg\forall XA^e}$$

$$\frac{\dots A(n)^\tau \dots}{\forall xA^\tau}\ \bigwedge\nolimits_{\forall xA^\tau}$$

Formulation of Ω^+-Rule

Definition

BI^{Ω^+}: $\mathrm{BI}_0^{\Omega^+}$ with $\Omega^+, \widetilde{\Omega}^+, Cut$.
$|\Omega^+|$ = the set of $q = < d, X >$ with $\mathrm{BI}_0^{\Omega^+} \ni d \vdash A(X)^i, \Delta_q^e$ where all formulas $B^i \in \Delta_q^e$ are arithmetical.

$$\frac{\ldots \Delta_q^e \ldots q \in |\Omega^+|}{\neg\forall XA^i}\ \Omega^+_{\neg\forall XA} \qquad !Y!\ \frac{A(Y)^i \quad \ldots \Delta_q^e \ldots q \in |\Omega^+|}{\emptyset}\ \widetilde{\Omega}^+_{\neg\forall XA}$$

$$\frac{C^i \quad \neg C^i}{\emptyset}$$

Remark: $\mathrm{BI}_0^{\Omega^+}$ contains $\bigvee_{\neg\forall XA^e}$. Hence, Δ_q^e may contain $\exists XA^e$ while Δ_q is just arithmetical in Buchholz's original definition.

Theorem (One-Step Reduction)

There is an operator $\mathscr{R}_C$ on derivations in BI^{Ω^+} such that if $d_0 \vdash_m \Gamma^e, \Pi^i, C^i$, $d_1 \vdash_m \Gamma^e, \Pi^i, \neg C^i$ and $rk(C^i) \leq m$, then $\mathscr{R}_C(d_0, d_1) \vdash_m \Gamma^e, \Pi^i$.

Proof.

If $d_0 = \bigwedge_{\forall X B(X)^i}(d_{00})$ and $d_1 = \Omega^+(d_{1q})_{q \in |\Omega^+|}$, then

$$\mathscr{R}_C(d_0, d_1) := \widetilde{\Omega}^+(\mathscr{R}_C(d_{00}, d_1), \mathscr{R}_C(d_0, d_{1q}))_{q \in |\Omega^+|}. \qquad \square$$

Predicative Cut-Elimination Operator $\mathscr{E}$

By applying $\mathscr{R}_C$, predicative cuts are eliminated:

Theorem (Cut-Reduction)

There is an operator $\mathscr{E}$ on derivations in BI^{Ω^+} such that if $d \vdash_{m+1} \Gamma^e, \Pi^i$, then $\mathscr{E}(d) \vdash_m \Gamma^e, \Pi^i$.

Proof. Familiar iteration of $\mathscr{R}_C$. □

Let $\mathscr{E}^m$ be m-times application of $\mathscr{E}$.

Theorem (Predicative C.E.)

if $d \vdash_{m+1} \Gamma^e, \Pi^i$, then $\mathscr{E}^m(d) \vdash_0 \Gamma^e, \Pi^i$.

Remark: $d \vdash_0 \Gamma^e, \Pi^i$ may contain some impredicative cuts $\widetilde{\Omega}^+$.

Collapsing Operator $\mathscr{D}$

- A sequent Γ is called *almost explicit* if all i-marked formulas $A^i \in \Gamma$ are arithmetical.
- Now we define a collapsing operator $\mathscr{D}$ for (not only arithmetical but) arbitrary almost explicit sequent.

Theorem (Collapsing)

There is an operator $\mathscr{D}$ such that if $BI^{\Omega^+} \ni d \vdash_0 \Gamma$ where Γ is almost explicit, then $BI_0^{\Omega^+} \ni \mathscr{D}(d) \vdash_0 \Gamma$.

Proof. $last(d) :=$ the last inference symbol of d.
If $last(d) = \bigvee^T_{\neg\forall X A^e}$, set $\mathscr{D}(d) := \bigvee^T_{\neg\forall X A^e}(\mathscr{D}(d_0))$.

Remark: $\bigvee^T_{\neg\forall X A^e}$ is included in $\mathrm{BI}_0^{\Omega^+}$ unlike Buchholz's treatment.

Collapsing Operator $\mathscr{D}$

Another important case: $last(d) = \widetilde{\Omega}^{+}(d_i)_{i \in \{0\} \cup |\Omega^{+}|}$.

Consider the leftmost premise: $d_0 \vdash_0 \Gamma^e, \Pi^i, A(Y)^i$. By IH, we get $\mathrm{BI}_0^{\Omega^+} \ni \mathscr{D}(d_0) \vdash \Gamma^e, \Pi^i, A(Y)^i$. Now $\mathscr{D}(d_0)$ is regarded as "input" for $\widetilde{\Omega}^{+}$. Formally, define $q_0 := (\mathscr{D}(d_0), Y)$, then $q_0 \in |\Omega^{+}|$.

For the input $\mathscr{D}(d_0)$, there should be an immediate subderivation d_{q_0} of $\widetilde{\Omega}^{+}$ s. t. $d_{q_0} \vdash_0 \Gamma^e, \Pi^i$.
Finally, apply IH to this d_{q_0}:

$$\mathscr{D}(d) := \mathscr{D}(d_{q_0}) \in \mathrm{BI}_0^{\Omega^+}. \ \square$$

$$\dfrac{\overset{\vdots}{\Gamma^e, \Pi^i, A(Y)^i} \quad \ldots \overset{\{q : \Delta^e_q, A(Y)^i\}}{\overset{\vdots}{\Gamma^e, \Pi^i, \Delta^e_q}} \ldots \quad q \in |\Omega^{+}|}{\Gamma^e, \Pi^i, A(Y)^i}\ \widetilde{\Omega}^{+}_{\neg\forall XA} \quad \Rightarrow \quad \dfrac{\overset{\{q_0 : \Gamma^e, \Pi^i, A(Y)^i\}}{\overset{\vdots}{\Gamma^e, \Pi^i}}}{\Gamma^e, \Pi^i}\ \mathscr{D}$$

Complete Cut-Elimination Theorem for BI^{Ω^+}

- By combining theorems obtained, we get the complete cut-elimination theorem for BI^{Ω^+}:

Theorem (Complete Cut-Elimination for BI^{Ω^+})

If $BI^{\Omega^+} \ni d \vdash_m \Gamma$ where Γ is almost explicit, then $BI_0^{\Omega^+} \ni \mathscr{D}(\mathscr{E}^m(d)) \vdash_0 \Gamma$.

Proof. By Predicative Cut-Elimination and Collapsing Theorems. □

Embedding of Π^1_1-CA via the Ω^+-Rule

- Following Buchholz, we embed Π^1_1-CA to BI^{Ω^+}.
- Difference from Buchholz: only implicit Π^1_1-CA is translated into the Ω^+-rule (explicit Π^1_1-CA: preserved).
- BI: parameter-free Π^1_1-CA with Schütte's ω-rule.
- Marking function m: assigning e or i to formulas A.

Theorem (Embedding)

If $d \in BI$, then $d^{\infty(m)} \vdash_{dg(d)} \Gamma(d)^m$ for any marking function m.

Proof.
If $d = (\bigvee_{\neg\forall XA}(d_0))$ and $m(\neg\forall XA) = \neg\forall XA^e$, set

$$d^{\infty(m)} := (\bigvee_{\neg\forall XA^e}(d_0^{\infty(m)})).$$

Otherwise:

$$d^{\infty(m)} := \Omega^+_{\neg\forall XA}(\mathscr{R}_{A(T)}(\mathscr{S}^X_T(d_q), d_0^{\infty(m[A(T)/i])}))_{q\in|\Omega^+|}. \quad \square$$

Complete Cut-Elimination Theorem for BI

- $\overrightarrow{e}$: the marking function assigning e to each formula A in L.
- d^* : the result of deleting all marks in sequents and inference rules of d.

Theorem (Complete Cut-Elimination for BI)

If $d \in BI$, then $BI \ni (\mathscr{D}(\mathscr{E}^n(d^{\infty(\overrightarrow{e})})))^* \vdash_0 \Gamma$ for some n.

Proof. By Embedding and C.C.E. for BI^{Ω^+} Theorems. Note that the inference rules in $\mathrm{BI}_0^{\Omega^+}$ become ones of BI after deleting marks. □

Remark: it is possible to iterate the Ω^+-rule (Akiyoshi, 2011).

Some Problems

- Precise game theoretic explanation of the Ω-rule?
- Extension of the Ω-rule to stronger system?
- Analogy with Girard's extension of computability predicate(candidates of reducibility)?