Vaught's conjecture in computability theory.

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The main theorem

Theorem ([M. 12])

Vaught's conjecture is equivalent to a statement in computability theory.

The full statement of the main theorem

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Theorem ([Burgess 78])

Let E be an analytic equivalence relation on $\mathbb{R}$. Then E has either countably many, $\aleph_1$ many, or perfectly many E-equivalence classes.

Recall: E has *perfectly many classes* if there is a perfect tree all whose paths are E-inequivalent.

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- So $|\{\text{models of } T\}| \leq \aleph_1$.

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Definition

An axiomatizable class $\mathbb{K}$ of countable structures is a

counterexample to Vaught's conjecture

if it has uncountably many models but not perfectly many.

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Topological Vaught's conjecture:
Consider a continuous action of a Polish group on a Polish space.
Any Borel invariant set has either countably many orbits or perfectly many.

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Def: *A is Turing equivalent to B*, written $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$.

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In general: Consider $\mathcal{A} = (A, c_0^A, c_1^A, ... f_0^A, f_1^A, ... R_0^A, R_1^A ...)$ where

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Note: A single structure can have many isomorphic presentations.

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Example

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- $X = \{n \in \mathbb{N} : \varphi(n)\}$, where φ is a computable infinitary formula.
  (these are $L_{\omega_1,\omega}$ formulas where disjunctions and conjunctions are computable)

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Spector's Theorem – hyperarithmetic-is-recursive

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Recall:
By *hyperarithmetic well-ordering* we mean
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  Again, for $\alpha < \omega_1^{CK}$, for any two lin.ord $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathbb{Z}^\alpha \cdot \mathcal{L}_1 \equiv_\alpha \mathbb{Z}^\alpha \cdot \mathcal{L}_2$.