Analytic equivalence relations with $\aleph_1$-many classes: A computability theoretic approach.

Antonio Montalbán

U.C. Berkeley

Sendai Logic School
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- $X \in L(\omega_1^{CK})$.
- $X = \{n \in \mathbb{N} : \varphi(n)\}$, where φ is a computable infinitary formula.

  (these are $L_{\omega_1,\omega}$ formulas where disjunctions and conjunctions are computable)

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Theorem:[Spector 1955] Every hyperarithmetic well ordering is isomorphic to a computable one.

Our main result

Definition:

- Given linear orderings $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ *embeds in* $\mathcal{B}$ if there is a strictly increasing map $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$. We write $\mathcal{A} \preccurlyeq \mathcal{B}$.
- $\mathcal{A}$ and $\mathcal{B}$ are *equimorphic* if $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$. We write $\mathcal{A} \sim \mathcal{B}$.

Example:

$$\omega + \omega^* + \omega + \omega^* + \cdots \quad \sim \quad \omega^* + \omega + \omega^* + \omega + \cdots$$

Observation: If α is an ordinal and $\mathcal{L} \sim \alpha$, then $\mathcal{L}$ is isomorphic to α.

Proof: $\mathcal{L} \preccurlyeq \alpha \implies \mathcal{L}$ is an ordinal and $\mathcal{L} \leq \alpha$.
$\alpha \preccurlyeq \mathcal{L} \implies \alpha \leq \mathcal{L}$ and hence $\mathcal{L} \cong \alpha$. □

Theorem: Every hyperarithmetic linear ordering is equimorphic to a recursive one.

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The proof uses Laver's theorem on the well-quasi-orderness of linear orderings to analyze their structure under embeddability.

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The proof uses Ulm invariants, and bi-embeddability invariants defined by [Barwise, Eklof 71]. It also uses that hyperarithmetic groups have Ulm rank $\leq \omega_1^{CK}$.

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- isomorphism on models of a *counterexample to Vaught's conjecture* (relativized);

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- T is a counterexample to Vaught's conjecture.
- $Mod(T)$ is linearly ordered by Muchnik reducibility on a cone.
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Obs: Since, in computability theory, almost all proofs relativize:
For "natural" classes of structures $\mathbb{K}$,

$\mathbb{K}$ *satisfies hyperarithmetic-is-recursive* $\iff$ *it does on a cone.*

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Question: What makes an equivalence relation satisfy hyperarithmetic-is-recursive on a cone?

A sufficient condition for hyp-is-rec.

Def: For $\mathfrak{K} \subseteq 2^\omega$, $(\mathfrak{K}, \equiv, r)$ is a *ranked equivalence relation* if
$\equiv$ is an equivalence relation on $\mathfrak{K}$, and $r \colon \mathfrak{K}/\equiv \to \omega_1$.

Def: $(\mathfrak{K}, \equiv, r)$ is *scattered* if
$r^{-1}(\alpha)$ contains countably many equivalence classes for each $\alpha \in \omega_1$.

Def: $(\mathfrak{K}, \equiv, r)$ is *projective* if
$\mathfrak{K}$ and $\equiv$ are projective and r has a projective presentation $2^\omega \to 2^\omega$.

Theorem ([M.] (ZFC+PD))

Let $(\mathfrak{K}, \equiv, r)$ be scattered projective ranked equivalence relation such that $\forall Z \in \mathfrak{K},\ r(Z) < \omega_1^Z$.
For every X on a cone, (i.e. $\exists Y \forall X \geq_T Y$,) every equivalence class with an X-hyperarithmetic member has an X-computable member.

Lemma: [Martin] (ZFC+PD) If $f \colon 2^\omega \to \omega_1$ is projective and $f(X) < \omega_1^X$,
then f is constant on a cone.

The main theorem

Theorem ([M.] (ZFC + ($0^\sharp$ exists) + $\neg$CH))

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This theorem applies to all the examples mentioned before.

Examples:

- isomorphism on well-orderings;
- bi-embeddability on linear orderings;
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In all the natural examples, the base of the cone is 0, which doesn't follow from the Theorem.

Removing ($0^\sharp$ exists) and ($\neg$CH)

Def: $\mathcal{S} \subseteq 2^{\mathbb{N}}$ is *cofinal (in the Turing degrees)* if $\forall Y\ \exists X \geq_T Y\ (X \in \mathcal{S})$.

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Theorem [M.] (ZF) The following are equivalent:

1. Every Σ^1_1-equivalence relation without perfectly many classes satisfies hyperarithmetic-is-recursive on a cone.
2. $0^\sharp$ exists.

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Some results about FF reductions

Theorem ([Fokina, Friedman, Harizanov, Knight, McCoy, M.])

The following classes of structures are on top (under $\leq_{eff}$):

- *trees,*
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A *nice class of structures* $\mathbb{K}$ is one axiomatizable by an $L_{\omega_1,\omega}$ sentence.

One direction of the question.

Theorem ([Knight–M. 12; Becker 12])

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Obs: The theorem above implies the downward direction.

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Theorem : [M.14] (ZFC+PD) If $\mathbb{K}$ is on top, there is a Δ^1_1 $g\colon \omega_1^{CK} \to \omega_1^{CK}$ such that there is a computable g-α-tree for $\mathbb{K}$ $(\forall \alpha < \omega_1^{CK})$.

The Main results

Theorem ([M. 12] (ZFC+$0^\sharp$ exists))

Let E be an analytic equivalence relation. The following are equivalent:

- *E does not have perfectly many classes.*
- *E satisfies hyperarithmetic-is-recursive on a cone.*
- *The classes of E are linearly ordered by Muchnik reducibility.*

Theorem [M.] Let $\sim$ be a degree-invariant, $\mathbf{\Sigma}^1_1$ equivalence relation. Exactly one of the following holds:

- $\sim$ has perfectly many classes on every cone.
- $\sim$ is trivial on a cone (i.e., $X \sim Y$ for all X, Y on some cone).
- $X \sim Y \iff \omega_1^X = \omega_1^Y$ for every X, Y on some cone.

Open question: Are the following equivalent?

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