On preserving AD by forcing

Daisuke Ikegami
(Shibaura Institute of Technology)

Joint work with Nam Trang

From now on...

We work in ZF+$\mathsf{DC}_{\mathbb{R}}$.

The choice priciple $\mathsf{DC}_{\mathbb{R}}$ states the following:

For any $A \subseteq \mathbb{R} \times \mathbb{R}$, if $(\forall x \in \mathbb{R})\ (\exists x \in \mathbb{R})\ (x, y) \in A$,
then $(\exists f : \omega \to \mathbb{R})\ (\forall n \in \omega)\ \big(f(n), f(n+1)\big) \in A$.

From now on...

We work in ZF+$\mathrm{DC}_{\mathbb{R}}$.

The choice priciple $\mathrm{DC}_{\mathbb{R}}$ states the following:

For any $A \subseteq \mathbb{R} \times \mathbb{R}$, if $(\forall x \in \mathbb{R})\ (\exists x \in \mathbb{R})\ (x, y) \in A$,

then $(\exists f : \omega \to \mathbb{R})\ (\forall n \in \omega)\ \big(f(n), f(n+1)\big) \in A$.

Let V denote the class of all sets.

Main Open Question

Question

Assume AD. Then can one find a set generic extension $V[G]$ of V with the following two properties?

1. $V[G]$ has more reals than V, and
2. $V[G]$ also satisfies AD.

Main Open Question

Question

Assume AD. Then can one find a set generic extension $V[G]$ of V with the following two properties?

1. $V[G]$ has more reals than V, and
2. $V[G]$ also satisfies AD.

The Axiom of Determinacy (AD) states that one of the players has a winning strategy for any Gale-Stewart game with a payoff set as a subset of the Baire space ω^ω.

Background: AD and properties of sets of reals

Theorem (Banach-Mazur, Davis, Mycielski-Steinhaus)

Assume AD. Then every set of reals is Lebesgue measurable, has the Baire property & the perfect set property.

Background: Proof for the Baire property

Definition

X: a topological space, $A \subseteq X$

1. A is nowhere dense if $\mathrm{int}\left(\mathrm{cl}(A)\right) = \emptyset$.

Background: Proof for the Baire property

Definition

X: a topological space, $A \subseteq X$

1. A is nowhere dense if $\text{int}\left(\text{cl}(A)\right) = \emptyset$.
2. A is meager if A is a countable union of nowhere dense sets.

Background: Proof for the Baire property

Definition

X: a topological space, $A \subseteq X$

1. A is nowhere dense if $\mathrm{int}\left(\mathrm{cl}(A)\right) = \emptyset$.
2. A is meager if A is a countable union of nowhere dense sets.
3. A has the Baire property if there is an open set U such that the symmetric difference between A and U $\left(\text{i.e., } (A \setminus U) \cup (U \setminus A)\right)$ is meager.

Background: Proof for the Baire property

Definition

X: a topological space, $A \subseteq X$

1. A is nowhere dense if $\mathrm{int}\left(\mathrm{cl}(A)\right) = \emptyset$.
2. A is meager if A is a countable union of nowhere dense sets.
3. A has the Baire property if there is an open set U such that the symmetric difference between A and U $\left(\text{i.e., } (A \setminus U) \cup (U \setminus A)\right)$ is meager.

Theorem (Banach-Mazur)

AD implies that every set of reals has the Baire property.

Proof: Whiteboard?

Back to Main Open Question

Question

Assume AD. Then can one find a set generic extension $V[G]$ of V with the following two properties?

1. $V[G]$ has more reals than V, and
2. $V[G]$ also satisfies AD.

Back to Main Open Question

Question

Assume AD. Then can one find a set generic extension $V[G]$ of V with the following two properties?

1. $V[G]$ has more reals than V, and
2. $V[G]$ also satisfies AD.

Example

If $V[G]$ is a generic extension of V via Cohen forcing, then in $V[G]$, the set $\omega^\omega \cap V$ does NOT have the Baire property.

Back to Main Open Question

Question

Assume AD. Then can one find a set generic extension $V[G]$ of V with the following two properties?

1. $V[G]$ has more reals than V, and
2. $V[G]$ also satisfies AD.

Example

If $V[G]$ is a generic extension of V via Cohen forcing, then in $V[G]$, the set $\omega^\omega \cap V$ does NOT have the Baire property.

In particular, AD fails in $V[G]$, so Cohen forcing destroys AD.

Main Result

Theorem (I., Trang)

Assume the following in V: ZF+ $\mathrm{DC}_{\mathbb{R}}$ + AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel".

If $\Theta^{V[G]} > \Theta^{V}$, then AD fails in $V[G]$.

Main Result

Theorem (I., Trang)

Assume the following in V: ZF+ $\mathrm{DC}_{\mathbb{R}}$ + AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel".

If $\Theta^{V[G]} > \Theta^V$, then AD fails in $V[G]$.

Definition

- $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ states that V is the smallest transitive proper class model of ZF containing all the sets of reals.

Main Result

Theorem (I., Trang)

Assume the following in V: ZF+ $\mathrm{DC}_{\mathbb{R}}$ + AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel".

If $\Theta^{V[G]} > \Theta^{V}$, then AD fails in $V[G]$.

Definition

- $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ states that V is the smallest transitive proper class model of ZF containing all the sets of reals.
- A set of reals is ∞-Borel if it is λ-Borel for some λ.

Main Result

Theorem (I., Trang)

Assume the following in V: ZF$+$ $\mathsf{DC}_{\mathbb{R}}$ $+$ AD $+$ $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ $+$ "Every set of reals is ∞-Borel".

If $\Theta^{V[G]} > \Theta^{V}$, then AD fails in $V[G]$.

Definition

- $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ states that V is the smallest transitive proper class model of ZF containing all the sets of reals.
- A set of reals is ∞-Borel if it is λ-Borel for some λ.
- The ordinal Θ is defined as follows:

$$\Theta = \sup\{\gamma \mid \gamma \text{ is a surjective image of } \mathbb{R}\}$$

Three remarks on the main result

Remark

1. The assumption "$V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" is essential, i.e., one can find a counter example of the theorem without this assumption.

Three remarks on the main result

Remark

1. The assumption "$V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" is essential, i.e., one can find a counter example of the theorem without this assumption.
2. The assumption "Every set of reals is ∞-Borel" is non-trivial.

Three remarks on the main result

Remark

1. The assumption "$V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" is essential, i.e., one can find a counter example of the theorem without this assumption.
2. The assumption "Every set of reals is ∞-Borel" is non-trivial.
3. In ZFC, $\Theta = (2^{\aleph_0})^+$, but under AD, Θ is quite large; it is a limit of measurable cardinals.

Background: Large cardinals

Let κ be an uncoutable cardinal.

Large cardinal properties of κ are often described as the existence of elementary embeddings $j\colon V \to M$ with $\kappa = \mathrm{crit}(j)$.

E.g., measurable cardinals

Background: Large cardinals

Let κ be an uncoutable cardinal.

Large cardinal properties of κ are often described as the existence of elementary embeddings $j\colon V \to M$ with $\kappa = \mathrm{crit}(j)$.

E.g., measurable cardinals

The closer M is to V, the stronger the large cardinal properties of κ are.

Background: Large cardinals

Let κ be an uncoutable cardinal.

Large cardinal properties of κ are often described as the existence of elementary embeddings $j\colon V \to M$ with $\kappa = \mathrm{crit}(j)$.

E.g., measurable cardinals

The closer M is to V, the stronger the large cardinal properties of κ are.

Question: What if $M = V$, i.e., $j\colon V \to V$?

Background: Kunen's Theorem

Theorem (Kunen)

There is NO elementary embedding $j\colon V \to V$ with $j \neq \mathrm{id}$ and $(V, \in, j) \models \mathrm{ZFC}$.

Proof: In whiteboard?

Motivation of Main Open Question

Kunen: There is NO non-trivial & elementary $j: V \to V$ such that $(V, \in, j) \models$ ZFC.

Open Question: How about replacing ZFC above with ZF only?

Motivation of Main Open Question

Kunen: There is NO non-trivial & elementary $j \colon V \to V$ such that $(V, \in, j) \models$ ZFC.

Open Question: How about replacing ZFC above with ZF only?

Hamkins et.al.: There is NO non-trivial & elementary $j \colon V \to V[G]$ such that $(V[G], \in, j) \models$ ZFC, where $V[G]$ is a set generic extension of V.

Woodin???: It is consistent to have $j \colon V \to V[G]$ as above if one demands $(V[G], \in, j) \models$ ZF only.
But in Woodin's example, $j \restriction \mathrm{Ord} = \mathrm{id}$.

Motivation of Main Open Question

Kunen: There is NO non-trivial & elementary $j\colon V \to V$ such that $(V, \in, j) \vDash$ ZFC.

Open Question: How about replacing ZFC above with ZF only?

Hamkins et.al.: There is NO non-trivial & elementary $j\colon V \to V[G]$ such that $(V[G], \in, j) \vDash$ ZFC, where $V[G]$ is a set generic extension of V.

Woodin???: It is consistent to have $j\colon V \to V[G]$ as above if one demands $(V[G], \in, j) \vDash$ ZF only.
But in Woodin's example, $j \restriction \mathrm{Ord} = \mathrm{id}$.

Question

How about demanding $(V[G], \in, j) \vDash \mathrm{ZF}+\mathrm{AD}$ and $V[G]$ has more reals than V?

Motivation of Main Open Question ctd.

Theorem (Woodin)

Assume AD. If M is an inner model of ZF and V has more reals than M, then ω_1^M must be countable.

Corollary

If $j\colon V \to V[G]$ is non-trivial & elementary, and $V[G]$ is a model of AD and has more reals than V, then $\text{crit}(j) = \omega_1^V$.

Motivation of Main Open Question ctd.

Theorem (Woodin)

Assume AD. If M is an inner model of ZF and V has more reals than M, then ω_1^M must be countable.

Corollary

If $j\colon V \to V[G]$ is non-trivial & elementary, and $V[G]$ is a model of AD and has more reals than V, then $\mathrm{crit}(j) = \omega_1^V$.

Coming back to Main Open Question:

Question

Assume AD. Then can one find a set generic extension $V[G]$ of V with the following two properties?

1. $V[G]$ has more reals than V, and
2. $V[G]$ also satisfies AD.

The main result stated again

Theorem (I., Trang)

Assume the following in V: ZF+ $\mathrm{DC}_{\mathbb{R}}$ + AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel".

If $\Theta^{V[G]} > \Theta^{V}$, then AD fails in $V[G]$.

Sketch of proof of the main result

For the proof, we use the following fact:

Fact (Woodin)

Assume AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel". Then HOD $= \mathrm{L}[X]$ for some $X \subseteq \Theta$ and V is definable in a set generic extension of HOD.

Sketch of proof of the main result

For the proof, we use the following fact:

Fact (Woodin)

Assume AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel". Then $\mathrm{HOD} = \mathrm{L}[X]$ for some $X \subseteq \Theta$ and V is definable in a set generic extension of HOD.

Sketch of proof:

Let G be P-generic over V and assume $V[G]$ was a model of AD.

Sketch of proof of the main result

For the proof, we use the following fact:

Fact (Woodin)

Assume AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel". Then HOD $= \mathrm{L}[X]$ for some $X \subseteq \Theta$ and V is definable in a set generic extension of HOD.

Sketch of proof:

Let G be P-generic over V and assume $V[G]$ was a model of AD.

Then $\Theta^{V[G]} > \Theta^V$ and $\Theta^{V[G]}$ is a limit of measurables in $V[G]$.

Sketch of proof of the main result

For the proof, we use the following fact:

Fact (Woodin)

Assume AD $+$ $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ $+$ "Every set of reals is ∞-Borel". Then HOD $= \mathrm{L}[X]$ for some $X \subseteq \Theta$ and V is definable in a set generic extension of HOD.

Sketch of proof:

Let G be P-generic over V and assume $V[G]$ was a model of AD.

Then $\Theta^{V[G]} > \Theta^V$ and $\Theta^{V[G]}$ is a limit of measurables in $V[G]$.

Now one can show that $X^\#$ exists in $V[G]$ for the X in the above fact, and so $X^\# \in V$.

Sketch of proof of the main result

For the proof, we use the following fact:

Fact (Woodin)

Assume AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel". Then HOD $= \mathrm{L}[X]$ for some $X \subseteq \Theta$ and V is definable in a set generic extension of HOD.

Sketch of proof:

Let G be P-generic over V and assume $V[G]$ was a model of AD.

Then $\Theta^{V[G]} > \Theta^V$ and $\Theta^{V[G]}$ is a limit of measurables in $V[G]$.

Now one can show that $X^\#$ exists in $V[G]$ for the X in the above fact, and so $X^\# \in V$.

But V is definable in a set generic extension of HOD.

Sketch of proof of the main result

For the proof, we use the following fact:

Fact (Woodin)

Assume AD + $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ + "Every set of reals is ∞-Borel". Then HOD $= \mathrm{L}[X]$ for some $X \subseteq \Theta$ and V is definable in a set generic extension of HOD.

Sketch of proof:

Let G be P-generic over V and assume $V[G]$ was a model of AD.

Then $\Theta^{V[G]} > \Theta^{V}$ and $\Theta^{V[G]}$ is a limit of measurables in $V[G]$.

Now one can show that $X^{\#}$ exists in $V[G]$ for the X in the above fact, and so $X^{\#} \in V$.

But V is definable in a set generic extension of HOD.

So $X^{\#} \in \mathrm{HOD} = \mathrm{L}[X]$, contradiction!

Coming back to the 1st remark

Remark (stated again)

In the main theorem, the assumption "$V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" is essential.

Coming back to the 1st remark

Remark (stated again)

In the main theorem, the assumption "$V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" is essential.

Fact (Solovay???)

It is consistent that AD holds, every set of reals is ∞-Borel, and there is a set of reals A which is NOT in the model $\mathrm{HOD}(\mathbb{R})$.

Coming back to the 1st remark

Remark (stated again)

In the main theorem, the assumption "$V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" is essential.

Fact (Solovay???)

It is consistent that AD holds, every set of reals is ∞-Borel, and there is a set of reals A which is NOT in the model $\mathsf{HOD}(\mathbb{R})$.

Example

Setting $M = \mathsf{HOD}(\mathbb{R})$ with $A \notin \mathsf{HOD}(\mathbb{R})$, M satisfies the assumptions of the Main Theorem except $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$, and there is a set generic extension $M[G]$ of M such that $M[G]$ satisfies AD while $\Theta^{M[G]} > \Theta^{M}$.

Coming back to the 1st remark

Remark (stated again)

In the main theorem, the assumption "$V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" is essential.

Fact (Solovay???)

It is consistent that AD holds, every set of reals is ∞-Borel, and there is a set of reals A which is NOT in the model $\mathsf{HOD}(\mathbb{R})$.

Example

Setting $M = \mathsf{HOD}(\mathbb{R})$ with $A \notin \mathsf{HOD}(\mathbb{R})$, M satisfies the assumptions of the Main Theorem except $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$, and there is a set generic extension $M[G]$ of M such that $M[G]$ satisfies AD while $\Theta^{M[G]} > \Theta^M$.

The P for $M[G]$ is a Vopenka-like forcing with $A \in M[G] \subseteq V$ such that $\mathbb{R}^M = \mathbb{R}^{M[G]}$. This is enough to guarantee the desired properties of $M[G]$.

Further results by other researchers

Theorem (Chan, Jackson)

Assume AD. Then

1. any non-trivial well-orderable forcing of length less than Θ destroys AD, and
2. if Θ is regular, then any non-trivial forcing which is a surjective image of $\mathbb{R}$ destroys AD.

Further results by other researchers

Theorem (Chan, Jackson)

Assume AD. Then

1. any non-trivial well-orderable forcing of length less than Θ destroys AD, and
2. if Θ is regular, then any non-trivial forcing which is a surjective image of $\mathbb{R}$ destroys AD.

The demanded properties of P to preserve AD while adding new reals if $V = \mathrm{L}(\mathcal{P}(\mathbb{R}))$:

1. P needs to collapse ω_1 while preserving Θ,
2. for any cardinal $\kappa < \Theta$ in V^P, NO club subset of κ in V witnesses the strong partition property of κ in V^P while there are unboundedly many strong parition property cardinals in Θ in V^P.

THE END.